The cyclicity of hyperbolic hemicycles

D. Marín[a,b] and J. Villadelprat[a]

[a] *Departament de Matemàtiques, Edifici Cc, Universitat Autònoma de Barcelona, 08193 Cerdanyola del Vallès (Barcelona), Spain*

[b] *Centre de Recerca Matemàtica, Edifici Cc, Campus de Bellaterra, 08193 Cerdanyola del Vallès (Barcelona), Spain*

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Abstract. In this paper we consider families of planar polynomial vector fields of degree n and study the cyclicity of a type of unbounded polycycle Γ called hemicycle. Compactified to the Poincaré disc, Γ consists of an affine straight line together with half of the line at infinity and has two singular points, which are hyperbolic saddles located at infinity. We prove four main results. Theorem A deals with the cyclicity of Γ when perturbed without breaking the saddle connections. For the other results we consider the case $n = 2$. More concretely they are addressed to the quadratic integrable systems belonging to the class Q_3^R and having two hemicycles, Γ_u and Γ_ℓ, surrounding each one a center. Theorem B gives the cyclicity of Γ_u and Γ_ℓ when perturbed inside the whole family of quadratic systems. In Theorem C we study the number of limit cycles bifurcating simultaneously from Γ_u and Γ_ℓ when perturbed as well inside the whole family of quadratic systems. Finally, in Theorem D we show that for three specific cases there exists a simultaneous alien limit cycle bifurcation from Γ_u and Γ_ℓ.

1 Introduction and main results

We begin by recalling the notion of limit periodic set as introduced in [22, Definition 10]. This is the fundamental object that we aim to study and its definition is given in terms of the Hausdorff topology, which for reader's convenience we briefly explain next.

Remark 1.1. Let S be a metrizable space and denote by $\mathcal{C}(S)$ the set of all compact non-empty subsets of S. Given any $K_1, K_2 \in \mathcal{C}(S)$ we define

$$d_H(K_1,K_2) = \sup_{x_1\in K_1, x_2\in K_2} \left\{ \inf_{x_2'\in K_2} d(x_1,x_2')\,,\ \inf_{x_1'\in K_1} d(x_1',x_2) \right\}.$$

One can readily show that d_H is a distance. It defines a topology on $\mathcal{C}(S)$, which is independent of the distance d chosen, that is called the *Hausdorff topology*. Moreover it turns out that

$$d_H(K_1,K_2) = \inf\left\{\varepsilon>0 : K_1 \subset N_\varepsilon(K_2) \text{ and } K_2 \subset N_\varepsilon(K_1)\right\},$$

where $N_\varepsilon(K)$ is the ε-neighbourhood of K. Finally, if (S,d) is a compact metric space then so is $(\mathcal{C}(S), d_H)$. The interested reader is referred to [20, p. 279] for both assertions. □

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Definition 1.2. A non-empty compact subset Γ of a surface S is a *limit periodic set* for a germ of a family $\{X_\mu\}_{\mu\approx\mu_0}$ of vector fields on S if there exists a sequence of parameters $\{\mu_n\}_n$ converging to μ_0 such that each X_{μ_n} has a limit cycle γ_n and the sequence $\{\gamma_n\}_n$ converges to Γ as $n\to\infty$ in the Hausdorff topology of the space $\mathcal{C}(S)$ of compact non-empty subsets of S. □

It is well known, see [22, Theorem 5], that any limit periodic set of a germ of an analytic family $\{X_\mu\}_{\mu\approx\mu_0}$ such that X_{μ_0} has only isolated singular points is either a singular point, a period orbit or a graphic of X_{μ_0}. We recall the notion of graphic and polycycle below:

Definition 1.3. Let X be a vector field on $\mathbb{R}^2$ (or $\mathbb{S}^2$). A *graphic* Γ for X is a compact, non-empty invariant subset which is a continuous image of $\mathbb{S}^1$ and consists of a finite number of isolated singularities $\{p_1,\dots,p_m,p_{m+1}=p_1\}$ (not necessarily distinct) and compatibly oriented separatrices $\{s_1,\dots,s_m\}$ connecting them (i.e., such that the α-limit set of s_j is p_j and the ω-limit set of s_j is p_{j+1}). A graphic is said to be *hyperbolic* if all its singular points are hyperbolic saddles. A *polycycle* is a graphic with a return map defined on one of its sides. □

The polycycles that we aim to study are unbounded and for this reason we need to compactify the vector field. Recall that to investigate the phase portrait of a polynomial vector field Y near infinity we can consider its Poincaré compactification $p(Y)$, see [1, §5] for details, which is an analytically equivalent vector field defined on the sphere $\mathbb{S}^2$. The points at infinity of $\mathbb{R}^2$ are in bijective correspondence with the points of the equator of $\mathbb{S}^2$, that we denote by ℓ_∞. Moreover the trajectories of $p(Y)$ in $\mathbb{S}^2$ are symmetric with respect to the origin and so it suffices to draw its flow in the closed northern hemisphere only, the so called Poincaré disc.

Definition 1.4. Let Π be an arbitrary collection of limit periodic sets for the germ of an analytic family $\{X_\mu\}_{\mu\approx\mu_0}$ of vector fields on $\mathbb{S}^2$. We define the *cyclicity* of Π with respect to $\{X_\mu\}_{\mu\approx\mu_0}$ as

$$\mathrm{Cycl}\big((\Pi,X_{\mu_0}),X_\mu\big):=\inf_{\varepsilon,\delta>0}\sup_{\mu\in B_\delta(\mu_0)}\#\big\{\gamma\text{ limit cycle of }X_\mu\text{ such that }d_H(\gamma,\Gamma)<\varepsilon\text{ for some }\Gamma\in\Pi\big\}.$$

□

Remark 1.5. Let us point out that if $\Pi=\{\Gamma\}$ then the cyclicity of Π coincides with the usual cyclicity $\mathrm{Cycl}\big((\Gamma,X_{\mu_0}),X_\mu\big)$ of the limit periodic set Γ, cf. [22, Definition 12]. In contrast, if Π consists of more than one limit periodic set then the cyclicity of Π accounts for the limit cycles bifurcating *simultaneously* from any of them. Finally, observe that if $\Pi\subset\Pi'$ then $\mathrm{Cycl}\big((\Pi,X_{\mu_0}),X_\mu\big)\leqslant\mathrm{Cycl}\big((\Pi',X_{\mu_0}),X_\mu\big)$. □

Note that the simultaneous cyclicity of $\{\Gamma_1,\dots,\Gamma_r\}$ may not coincide with the cyclicity of $\Gamma_1\cup\dots\cup\Gamma_r$, even in case that the latter is a limit periodic set. For instance, consider a germ $\{X_\mu\}_{\mu\approx\mu_0}$ such that X_{μ_0} has a saddle point with two homoclinic loops Γ_- and Γ_+ making up a "figure eight-loop" $\Gamma=\Gamma_-\cup\Gamma_+$, see Figure 1. Then the values of

$$\begin{array}{lll}\mathrm{Cycl}\big((\Gamma_-,X_{\mu_0}),X_\mu\big) & \mathrm{Cycl}\big((\Gamma_+,X_{\mu_0}),X_\mu\big) & \mathrm{Cycl}\big((\Gamma,X_{\mu_0}),X_\mu\big)\\ \mathrm{Cycl}\big((\{\Gamma_-,\Gamma_+\},X_{\mu_0}),X_\mu\big) & \mathrm{Cycl}\big((\{\Gamma_\pm,\Gamma\},X_{\mu_0}),X_\mu\big) & \mathrm{Cycl}\big((\{\Gamma_-,\Gamma_+,\Gamma\},X_{\mu_0}),X_\mu\big)\end{array}$$

may be all different. On the other hand, it is clear that

$$\max_{j\in\{1,2,\dots,r\}}\big\{\mathrm{Cycl}\big((\Gamma_j,X_{\mu_0}),X_\mu\big)\big\}\leqslant\mathrm{Cycl}\big((\{\Gamma_1,\dots,\Gamma_r\},X_{\mu_0}),X_\mu\big)\leqslant\sum_{j=1}^r\mathrm{Cycl}\big((\Gamma_j,X_{\mu_0}),X_\mu\big).$$

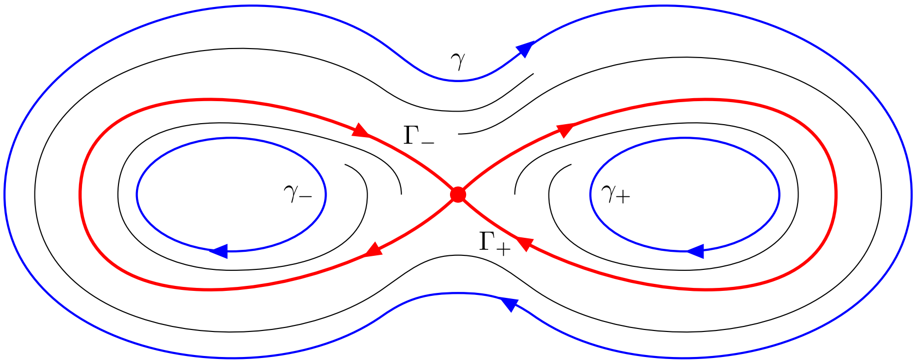

Figure 1: "Figure eight-loop" $\Gamma = \Gamma_- \cup \Gamma_+$ formed by two homoclinic connections Γ_- and Γ_+. The limit cycles γ_-, γ_+ and γ are close (with respect to the Hausdorff distance) to Γ_-, Γ_+ and Γ, respectively.

In the first part of the paper we consider the family of vector fields $\{X_\mu\}_{\mu\in\Lambda}$ given by

$$X_\mu \quad \begin{cases} \dot{x} = yf(x,y;\mu) + g(x;\mu), \\ \dot{y} = yq(x,y;\mu), \end{cases} \tag{1}$$

where Λ is an open subset of $\mathbb{R}^N$ and f, g and q are polynomials with the coefficients depending analytically on μ. We assume that $\deg(f) = \deg(q) = n$ and $\deg(g) = n+1$ and that the following hypothesis hold:

H1 $g(x;\mu) < 0$ for all $x \in \mathbb{R}$ and $\mu \in \Lambda$, which implies that n is odd, and

H2 $\ell_{n+1}(x,y;\mu) := yf_n(x,y;\mu) - xq_n(x,y;\mu) + g_{n+1}x^{n+1} > 0$ for all $(x,y) \neq (0,0)$ and $\mu \in \Lambda$.

Here, and in what follows, $f_n(x,y;\mu)$ and $q_n(x,y;\mu)$ denote, respectively, the homogeneous part of degree n of $f(x,y;\mu)$ and $q(x,y;\mu)$, whereas $g_{n+1}(\mu)$ is the leading coefficient of $g(x;\mu)$. The second hypothesis is related with the angle variation θ of the solutions of (1) near the infinity because one can verify that

$$r^2\dot{\theta} = y\big(xq(x,y) - yf(x,y) - g(x)\big).$$

Since ℓ_{n+1} is a homogeneous polynomial of even degree, **H2** is equivalent to $zf_n(1,z) - q_n(1,z) + g_{n+1} < 0$ and $f_n(z,1) - zq_n(z,1) + g_{n+1}z^{n+1} < 0$ for all $z \in \mathbb{R}$ and $\mu \in \Lambda$.

Conditions **H1** and **H2** guarantee that, after compactifying the polynomial vector field X_μ to the Poincaré disc, the boundary of the upper (respectively, lower) half-plane is a polycycle Γ_u (respectively, Γ_ℓ) with two hyperbolic saddles, see Figure 2,

$$s_1 := \{y = 0, x > 0\} \cap \ell_\infty \text{ and } s_2 := \{y = 0, x < 0\} \cap \ell_\infty.$$

This type of polycycle, formed by an invariant line and half of the equator of $\mathbb{S}^2$, is called *hemicycle* in [7]. Moreover the vector fields of the form (1) verifying **H1** and **H2** are called *D-systems* by the authors in [9].

Our first main result is addressed to the cyclicity of Γ_u when perturbed *inside* the family of D-systems. This result will be given in terms of two functions $d_1(\mu)$ and $d_2(\mu)$. In order to define them we first need

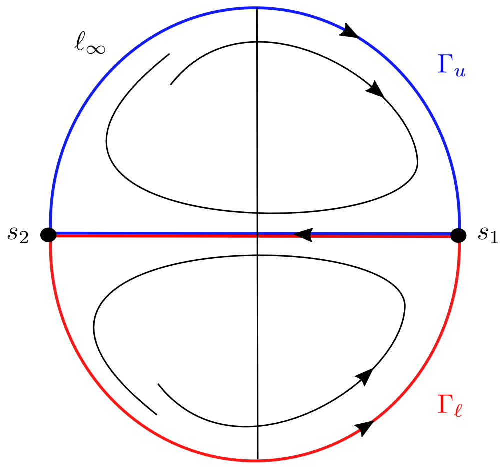

Figure 2: Placement of the hyperbolic saddles and the polycycles Γ_u and Γ_ℓ in the Poincaré disc for the polynomial vector field (1).

to introduce several other functions. For the sake of shortness we shall omit the dependence of μ in these functions when there is no risk of confusion. We define

$$K(x_1, x_2; \mu) := 1 - \left.\frac{xq(x,y)}{yf(x,y)+g(x)}\right|_{(x,y)=\left(\frac{1}{x_1},\frac{x_2}{x_1}\right)} \text{ and } \lambda(\mu) := -K(0,0;\mu) = -1 + \frac{q_n(1,0)}{g_{n+1}} > 0. \quad (2)$$

Let us remark that, on account of **H1** and **H2**, the functions K and $1/K$ are well defined in a neighbourhood of $\{x_1 = 0\}$ and $\{x_2 = 0\}$. Then, setting

$$M_1(u) = \exp\left(\int_0^u \left(\frac{1}{K(0,z)} + \frac{1}{\lambda}\right)\frac{dz}{z}\right)\partial_1\left(\frac{1}{K}\right)(0,u)$$

and

$$M_2(u) = \exp\left(\int_0^u (K(z,0)+\lambda)\frac{dz}{z}\right)\partial_2 K(u,0),$$

we define

$$F_1(\mu) = -\int_0^{+\infty}\Big(M_1(z) - M_1(0) + \exp(G_1)\big(M_1(-z) - M_1(0)\big)\Big)\frac{dz}{z^{1+1/\lambda}}, \quad (3)$$

$$F_2(\mu) = \int_0^{+\infty}\Big(M_2(-z) - M_2(0) + \exp(G_2)\big(M_2(z) - M_2(0)\big)\Big)\frac{dz}{z^{1+\lambda}} \quad (4)$$

and

$$F_3(\mu) = G_2\big(\partial_1 K \partial_2 K + \partial_{12} K\big)(0,0), \quad (5)$$

where

$$G_1 = \int_{-1}^{1}\left(\frac{q_n(1,z)}{\ell_{n+1}(1,z)} + 1 + \frac{1}{\lambda} + \frac{zq_n(z,1)}{\ell_{n+1}(z,1)}\right)\frac{dz}{z} \text{ and } G_2 = \int_0^{+\infty}\left(\frac{q(z,0)}{g(z)} + \frac{q(-z,0)}{g(-z)}\right)dz.$$

Taking this notation into account, the functions that determine the cyclicity (and stability) of the polycycle Γ_u at first and second order are the following:

$$d_0(\mu) := -\int_{-\infty}^{+\infty} \left(\frac{q(z,0)}{g(z)} + \lambda \frac{q_n(z,1)}{\ell_{n+1}(z,1)} \right) dz \;\text{ and }\; d_1(\mu) := \begin{cases} F_1(\mu) & \text{if } \lambda(\mu) > 1, \\ F_2(\mu) & \text{if } \lambda(\mu) < 1, \\ F_3(\mu) & \text{if } \lambda(\mu) = 1. \end{cases} \tag{6}$$

By Theorem 2.1, d_0 is analytic on Λ and d_1 is analytic on $\Lambda \setminus \Lambda_1$, where $\Lambda_1 := \{\mu \in \Lambda : \lambda(\mu) = 1\}$. In the statement $\mathscr{R}_u(\,\cdot\,;\mu)$ stands for the return map of the vector field X_μ around the polycycle Γ_u, see Figure 2, and we use the notion of functional independence given in Definition A.11.

Theorem A. *Let us consider the family of polynomial vector fields $\{X_\mu\}_{\mu\in\Lambda}$ given in (1) and verifying the assumptions* **H1** *and* **H2**. *Then the following assertions hold for any $\mu_0 \in \Lambda$ such that $\mathscr{R}_u(\,\cdot\,;\mu_0) \not\equiv Id$:*

(a) *If $d_0(\mu_0) \neq 0$ then* $\mathrm{Cycl}\big((\Gamma_u, X_{\mu_0}), X_\mu\big) = 0$.

(b) *If d_0 vanishes and is independent at μ_0 then* $\mathrm{Cycl}\big((\Gamma_u, X_{\mu_0}), X_\mu\big) \geqslant 1$.

(c) *If $d_1(\mu_0) \neq 0$ then* $\mathrm{Cycl}\big((\Gamma_u, X_{\mu_0}), X_\mu\big) \leqslant 1$.

(d) *If d_0 and d_1 vanish and are independent at μ_0 and $\lambda(\mu_0) \neq 1$ then* $\mathrm{Cycl}\big((\Gamma_u, X_{\mu_0}), X_\mu\big) \geqslant 2$. *Moreover the same lower bound holds in case that $\lambda(\mu_0) = 1$ and the restrictions $d_0|_{\Lambda_1}$ and $d_1|_{\Lambda_1}$ vanish and are independent at μ_0.*

With regard to the application of Theorem A it is worth noting that if $d_0(\mu_0) \neq 0$, or $d_1(\mu_0) \neq 0$, then $\mathscr{R}_u(\,\cdot\,;\mu_0) \not\equiv \mathrm{Id}$. This is a consequence of Theorem 2.1, which is a fundamental result to prove Theorem A.

The stability of this kind of hemicycle was previously studied in [9, Theorem 7]. Indeed, using our notation, the authors prove that $\mathscr{R}_u(s;\mu) = e^{d_0(\mu)}s + \mathrm{o}(s)$, so that if $d_0(\mu_0) < 0$ (respectively, $d_0(\mu_0) > 0$) then the polycycle Γ_u of the vector field X_{μ_0} is asymptotically stable (respectively, unstable). In this paper, by performing a second order analysis we also obtain the stability in case that $d_0(\mu_0) = 0$ and $d_1(\mu_0) \neq 0$ (see Remark 2.2). That being said, the goal of the present paper is not to study the stability of the hemicycle but its cyclicity. The first notion concerns single vector fields, whereas the second one is addressed to families of vector fields (i.e., depending on parameters). This is the reason why we need the remainder in the asymptotic expansion of $\mathscr{R}_u(s;\mu)$ at $s = 0$ to be uniform with respect to the parameters. Let us also note that similar results (for both, stability and cyclicity) can be obtained for the hemicycle Γ_ℓ by performing the change of variables $(x,y) \mapsto (x,-y)$.

Theorem A is a general result for the cyclicity of the polycycle Γ_u of a D-system X_{μ_0} with $\mathscr{R}_u(\,\cdot\,;\mu_0) \not\equiv \mathrm{Id}$ when perturbed inside the family of D-systems (1). Note that in doing so the polycycle Γ_u is persistent (i.e., the connections between the two vertices remain unbroken through the perturbation). In contrast, the rest of our main results concern the cyclicity of quadratic D-systems X_{μ_0} with $\mathscr{R}_u(\,\cdot\,;\mu_0) \equiv \mathrm{Id}$ when perturbed inside the whole family of quadratic systems. This means in particular that the connection breaks, see Figure 7. More concretely, in Theorems B, C and D, for each $(a_0, b_0) \in (-2,0) \times (0,2)$, we perturb the quadratic D-system

$$\begin{cases} \dot{x} = \frac{b_0-2}{4} + (1-b_0)y + a_0x^2 + b_0y^2, \\ \dot{y} = -2xy, \end{cases} \tag{7}$$

that one can show it verifies assumptions **H1** and **H2**. Moreover it has two centers, located at the points $(0, \frac{1}{2})$ and $(0, \frac{b_0-2}{2b_0})$ whose period annulus foliate, respectively, the upper and lower half-planes, see Figure 3.

Theorem B. *If $(a_0, b_0) \in (-2,0) \times (0,2)$ and $a_0 \neq -1$ then the cyclicities of Γ_u and Γ_ℓ when we perturb (7) inside the whole family of quadratic differential systems are exactly 2. Furthermore both cyclicities are at least 2 for $(a_0, b_0) \in \{-1\} \times (0,2)$.*

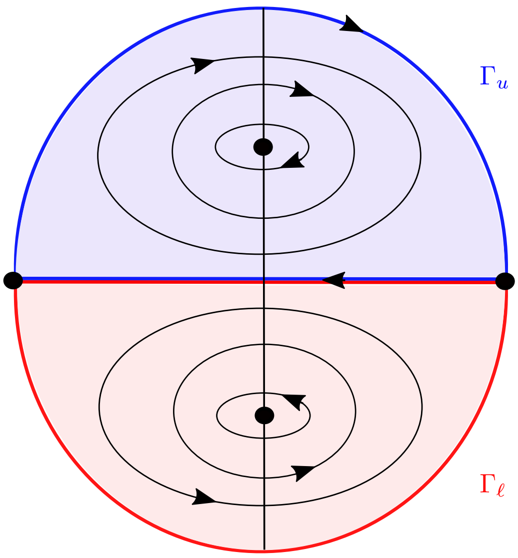

Figure 3: Phase portrait in the Poincaré disc of the quadratic differential system (7) for each $(a_0, b_0) \in (-2, 0) \times (0, 2)$.

Using the terminology from [12], both centers of the unperturbed system (7) are inside the reversible component Q_3^R of the *center manifold* of the quadratic systems. There are three other components: Hamiltonian Q_3^H, codimension four Q_4 and generalized Lotka-Volterra Q_3^{LV}. It turns out (c.f. Lemma 3.3) that the centers of the unperturbed system belong also to the Q_3^{LV} component in case that $(a_0+b_0)(a_0-b_0+2) = 0$, and when this occurs the proof of Theorem B is a little more difficult.

Closely related to Theorem B, it is to be quoted a result due to Swirszcz, see [29, Theorem 1]. Indeed, in that paper the author also studies the cyclicity of a polycycle of a quadratic reversible system when perturbed inside the whole quadratic family. More concretely, he perturbs the differential system (7) but taking $(a_0, b_0) \in \mathcal{S} := \{0 < b_0 < -a_0\} \cap \{a_0 < -2\}$. For these parameters the singular point $(0, \frac{1}{2})$ is also a center but the polycycle at the boundary of its period annulus is not an hemicycle. It is a bicycle Γ_b with the two vertices at infinity, and consisting of a branch of a hyperbola together with a segment of ℓ_∞. Recall that the *period annulus* of a center p is its largest punctured neighbourhood $\mathscr{P}$ which is entirely covered by periodic orbits, and that its boundary $\partial\mathscr{P}$ has two connected components: the center itself and a polycycle. By using a completely different approach than ours, and with a lower level of detail in the proofs, Swirszcz identifies a curve $\mathcal{C}$ (see Figure 4) such that the cyclicity of Γ_b is 3 if $(a_0, b_0) \in \mathcal{S} \cap \mathcal{C}$ and 2 if $(a_0, b_0) \in \mathcal{S} \setminus \mathcal{C}$. It is to be noted that the only parameter value in $\mathcal{S}$ which intersects another center component is $(a_0, b_0) = (-4, 2)$, that belongs also to the Q_4 component.

In another vein, Iliev studies in his seminal paper [12] the cyclicity of the period annulus $\mathscr{P}$ of the quadratic centers. We stress that the definition of cyclicity for $\mathscr{P}$ is different than the one for a polycycle because the former is open (see Definition 1.6). Among other results Iliev proves that the cyclicity of the period annulus $\mathscr{P}$ of the center at $(0, \frac{1}{2})$ of the differential system (7) is 3 for $(a_0, b_0) = (-4, 2)$ and 2 for $(a_0, b_0) = (-1, 1)$. These two parameters are denoted, respectively, by Q_4^+ and S_1 in Figure 4. Moreover he conjectures that the cyclicity of $\mathscr{P}$ is equal to 3 if (a_0, b_0) is inside the shaded area in Figure 4 and equal to 2 if (a_0, b_0) is outside. Previous to Iliev's conjecture, there is a result by Shafer and Zegeling (see [26, Theorem 3.2]) that determines some regions where the cyclicity of $\mathscr{P}$ is equal to 3. They also give a numerical approximation to the curve $\mathcal{C}$. In this setting Theorem B reinforces Iliev's conjecture because

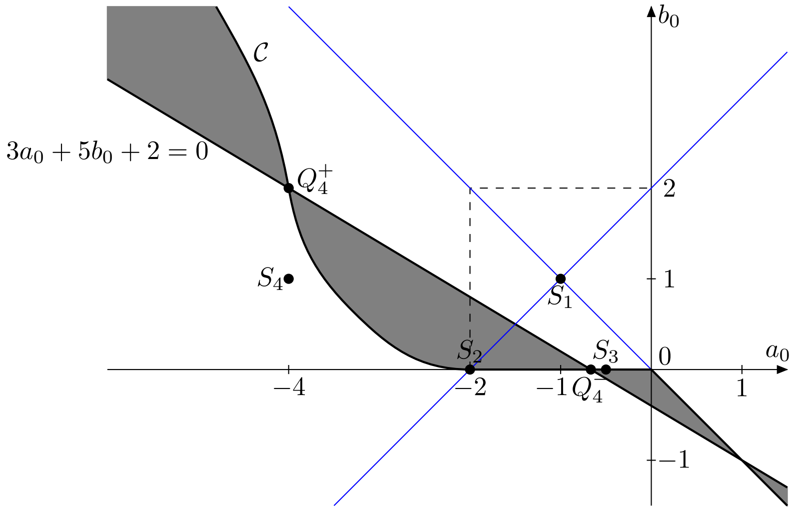

Figure 4: According to Illiev's conjecture, the shaded area corresponds to those parameters (a_0, b_0) for which the period annulus $\mathscr{P}$ of the center at $(0, \frac{1}{2})$ of system (7) has cyclicity 3. Its boundary has two components: the straight line $3a_0 + 5b_0 + 2 = 0$ and a piecewise curve $\mathcal{C}$. The straight line corresponds to parameters for which the center itself has cyclicity 3. The curve $\mathcal{C}$ corresponds to parameters for which the polycycle at $\partial\mathscr{P}$ has cyclicity 3. The parameters $S_1 = (-1, 1)$, $S_2 = (-2, 0)$, $S_3 = (-\frac{1}{2}, 0)$ and $S_4 = (-4, 1)$ are the four isochronous quadratic centers. The blue straight lines are the intersection points with the component Q_3^{LV} of the center manifold. The parameters $Q_4^+ = (-4, 2)$ and $Q_4^- = (-\frac{2}{3}, 0)$ are the intersection points with the component Q_4.

it shows that the curve $\mathcal{C}$ does not enter the square $(a_0, b_0) \in (-2, 0) \times (0, 2)$.

Let us recall at this point that *Hilbert's 16th problem* asks for the maximum number $H(n)$ of limit cycles of a planar polynomial differential system of degree $\leqslant n$. It is still open for any $n \geqslant 2$. In 1994 Dumortier, Roussarie and Rousseau conceived a program (see [7]) to prove that $H(2)$ is finite. In short, they reduced this problem to prove the finite cyclicity for only 121 (different classes of) graphics occurring in quadratic systems. According to the notation in that paper, the quadratic system (7) with $(a_0, b_0) \in (-2, 0) \times (0, 2)$ is inside the class H_2^1 of hyperbolic hemicycles surrounding a center (see [7, Figure 7]). Thus, Theorem B can be viewed as a contribution to the completion of the program to prove that $H(2) < \infty$. Nevertheless some authors (e.g. [24]) attribute to Mourtada the proof of the finite cyclicity of any hyperbolic polycycle in an unpublished series of manuscripts (see [15, Theorem 0] and references therein). For other results about the cyclicity of quadratic hemicycles in this context the interested reader is referred to [5, 23].

Note that Theorem B provides the cyclicity of Γ_u and Γ_ℓ individually, i.e., taking $\Pi = \{\Gamma_u\}$ and $\Pi = \{\Gamma_\ell\}$ in Definition 1.4. In our third main result we study the cyclicity of $\Pi = \{\Gamma_u, \Gamma_\ell\}$, c.f. Remark 1.5, when we perturb (7) inside the family of quadratic differential systems. In its statement we use the following parameter subsets:

$$\mathcal{K}_1 := \{(a_0, b_0) \in (-2, 0) \times (0, 2) : a_0 + b_0 \leqslant 0 \text{ or } a_0 - b_0 + 2 \leqslant 0\}$$

and

$$\mathcal{K}_2 := \{(a_0, b_0) \in (-2, 0) \times (0, 2) : a_0 + b_0 > 0 \text{ and } a_0 - b_0 + 2 > 0\}.$$

Theorem C. *If* (a_0,b_0) *belongs to* $\mathcal{K}_1\setminus\{a_0=-1\}$ (*respectively,* $\mathcal{K}_2$) *then the cyclicity of* $\Pi=\{\Gamma_u,\Gamma_\ell\}$ *when we perturb* (7) *inside the whole family of quadratic differential systems is exactly 3* (*respectively, 2*)*. Moreover it is at least 3 for* $(a_0,b_0)\in\{-1\}\times(0,2)$.

We stress that Theorem C deals with the simultaneous bifurcation of limit cycles from Γ_u and Γ_ℓ, which are the outer boundaries of two period annuli. The simultaneous bifurcation of limit cycles from the two period annuli has been studied for $a_0=-\frac{3}{2}$ and $a_0=-\frac{1}{2}$ in [14] and [3], respectively, and also for $(a_0,b_0)=(-\frac{1}{2},\frac{1}{2})$ and $(a_0,b_0)=(-1,1)$ in [21] and [8], respectively. The authors do not know of any previous work dealing with the simultaneous bifurcation from two polycycles.

Following Gavrilov [10] we introduce the notion of cyclicity of an open subset U as follows. He considers the case when U is a period annulus and here we extend it slightly.

Definition 1.6. Let $\{X_\mu\}_{\mu\approx\mu_0}$ be a germ of an analytic family of vector fields on $\mathbb{S}^2$ and let K be a compact subset of $\mathbb{S}^2$. We define the cyclicity of K with respect to the germ $\{X_\mu\}_{\mu\approx\mu_0}$ as

$$\mathrm{Cycl}_G\big((K,X_{\mu_0}),X_\mu\big)=\inf_{\varepsilon,\delta>0}\sup_{\mu\in B_\delta(\mu_0)}\#\big\{\gamma\subset N_\varepsilon(K)\text{ limit cycle of }X_\mu\big\}\in\mathbb{Z}_{\geq0}\cup\{\infty\},$$

where $N_\varepsilon(K)$ is the tubular ε-neighbourhood of K. If $U\subset\mathbb{S}^2$ is open we define

$$\mathrm{Cycl}_G\big((U,X_{\mu_0}),X_\mu\big)=\sup\big\{\mathrm{Cycl}_G\big((U,X_{\mu_0}),X_\mu\big):K\subset U\text{ compact}\big\}.$$

□

In case that U is a period annulus with finite cyclicity in the above sense, Gavrilov proves in [10, Theorem 1] that $\mathrm{Cycl}_G\big((U,X_{\mu_0}),X_\mu\big)$ is the same as in an appropriate one-parameter analytic deformation. This is related with the notion of essential perturbation introduced by Illiev [12] and enables to tackle the problem by computing Melnikov functions. This well-known approach allows to bound the number of limit cycles bifurcating from any compact set $K\subset U$ by means of the Weierstrass Preparation Theorem, however it gives not enough information on $U\setminus K$. This motivates the following definition.

Definition 1.7. Let $\{X_\mu\}_{\mu\approx\mu_0}$ be a germ of an analytic family of vector fields on $\mathbb{S}^2$ and consider an open subset U of $\mathbb{S}^2$. We define the *boundary cyclicity of U from inside* as

$$\underline{\mathrm{Cycl}}{}_G^U\big((\partial U,X_{\mu_0}),X_\mu\big)=\inf\big\{\mathrm{Cycl}_G\big((U\setminus K,X_{\mu_0}),X_\mu\big):K\subset U\text{ compact}\big\}.$$

□

If ∂U is a polycycle with a return map which is not the identity then it can be shown by a compactness and continuity argument that $\underline{\mathrm{Cycl}}{}_G^U\big((\partial U,X_{\mu_0}),X_\mu\big)=0$. On the other hand, we prove in Lemma 5.1 that

$$\underline{\mathrm{Cycl}}{}_G^U\big((\partial U,X_{\mu_0}),X_\mu\big)\leqslant\mathrm{Cycl}_G\big((\partial U,X_{\mu_0}),X_\mu\big).$$

These two facts lead to the definition of alien bifurcation that we propose in the present paper:

Definition 1.8. Let $\{X_\mu\}_{\mu\approx\mu_0}$ be a germ of an analytic family of vector fields on $\mathbb{S}^2$ such that X_{μ_0} is a D-system satisfying hypothesis **H1** and **H2**. Assume additionally that the return maps $\mathscr{R}_u(\,\cdot\,;\mu_0)$ and $\mathscr{R}_\ell(\,\cdot\,;\mu_0)$ of the hemicycles Γ_u and Γ_ℓ are both the identity. Taking $U=\mathbb{R}^2\setminus\{y=0\}$, if

$$\underline{\mathrm{Cycl}}{}_G^U\big((\partial U,X_{\mu_0}),X_\mu\big)<\mathrm{Cycl}_G\big((\partial U,X_{\mu_0}),X_\mu\big)\tag{8}$$

then we say that an *alien limit cycle bifurcation* occurs at $\partial U=\Gamma_u\cup\Gamma_\ell$ for $\{X_\mu\}_{\mu\approx\mu_0}$ from inside U. □

Under the hypothesis in Definition 1.8, the vertices of Γ_u and Γ_ℓ are hyperbolic saddles. In this case it follows from Lemma 5.2 that

$$\mathrm{Cycl}_G\big((\partial U, X_{\mu_0}), X_\mu\big) = \mathrm{Cycl}\big((\{\Gamma_u, \Gamma_\ell\}, X_{\mu_0}), X_\mu\big).$$

Hence Definition 1.8 takes into account the *simultaneous* bifurcation of limit cycles from Γ_u and Γ_ℓ. That being said, our notion of alien limit cycle bifurcation is defined verbatim in situations with a single hyperbolic polycycle Γ. For instance, if Γ is a saddle loop (or a bicycle) then the definition of alien bifurcation at Γ would be the same but taking U to be the bounded connected component of $\mathbb{R}^2 \setminus \Gamma$. Since the earliest paper by Dumortier and Roussarie [6], the term "alien limit cycle" has appeared in the literature (see [2, 4, 13, 27, 28, 30]) to describe a limit cycle bifurcating from a polycycle which can not detected as a zero of the Melnikov function of first order. Our definition differs from this one because, as we explain in Remark 5.3, the inequality in (8) holds if there is a limit cycle bifurcation which can not be detected by any Melnikov function of *any* order. In this regard we obtain the following result about alien bifurcations in the quadratic family:

Theorem D. *If $(a_0, b_0) \in \{(-1,1), (-\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, \frac{3}{2})\}$ then an alien limit cycle bifurcation occurs at $\Gamma_u \cup \Gamma_\ell$ when we perturb (7) inside the whole family of quadratic differential systems.*

Let us remark that in the present paper we consider families of planar polynomial vector fields $\{X_\mu\}_{\mu\in\Lambda}$ and that the statements of our main results should more formally be addressed to the compactified family $\{p(X_\mu)\}_{\mu\in\Lambda}$ of analytic vector fields on the Poincaré sphere $\mathbb{S}^2$. For simplicity in the exposition we commit an abuse of language by identifying both families. It is clear that the number of limit cycles of X_μ and $p(X_\mu)$ is the same because the line at infinity ℓ_∞ is invariant in all the cases under consideration. Related with this we note that, although the corresponding analytic extension of the polynomial vector field to $\mathbb{S}^2$ does not descend to the quotient $\mathbb{RP}^2$ of $\mathbb{S}^2$ by the central symmetry with respect to the origin, the induced foliation does. Since limit cycles depend on the foliation, and not on the specific way in which the orbits are parametrized, one could consider the notion of cyclicity in the real projective plane $\mathbb{RP}^2$ instead of the sphere $\mathbb{S}^2$. It is worth to point out that these two notions are not equivalent. Indeed, the two hemicycles Γ_u and Γ_ℓ in $\mathbb{S}^2$ project to the same polycycle $\bar{\Gamma}_u = \bar{\Gamma}_\ell$ on $\mathbb{RP}^2$ (see Figure 5) and by applying Theorems B and C, respectively,

$$\mathrm{Cycl}\big((\Gamma_u, X_{\mu_0}), X_\mu\big) = 2 \text{ and } \mathrm{Cycl}_{\mathbb{RP}^2}\big((\bar{\Gamma}_u, X_{\mu_0}), X_\mu\big) = 3$$

for any $(a_0, b_0) \in \mathcal{K}_1 \setminus \{a_0 = -1\}$.

The paper is organized as follows. Sections 2 and 3 are devoted to prove Theorems A and B, respectively. Both results strongly rely on the asymptotic development of the difference map $\mathscr{D}(s;\mu)$ given in Theorem 2.1. This is a rather technical result that follows by applying the tools developed in [16, 17, 18] to study the Dulac map and its proof is deferred to Appendix B for reader's convenience. Another important ingredient in the proof of Theorem B is Theorem 3.5, which provides a very useful division of the difference map in the ideal generated by its coefficients. The proofs of Theorems C and D are given in sections 4 and 5, respectively. Appendix A gathers the essential definitions and results from [16, 17, 18] that we use in the present paper, together with some other auxiliary results. Finally, in Appendix B we demonstrate Theorem 2.1 and Proposition 3.2, which have the longest and most technical proofs.

2 Proof of Theorem A

In this section we consider the family of vector fields $\{X_\mu\}_{\mu\in\Lambda}$ given by (1) and satisfying the hypothesis **H1** and **H2**. We take two local transverse sections, Σ_1 and Σ_2 parametrised, respectively, by $s \mapsto (0, \frac{1}{s})$ and $s \mapsto (0, s)$ with $s > 0$. We also define $D_+(s;\mu)$ to be the Dulac map of X_μ from Σ_1 to Σ_2 and $D_-(s;\mu)$ to be the Dulac map of $-X_\mu$ from Σ_1 to Σ_2, see Figure 6. The limit cycles of X_μ that are close to Γ_u in

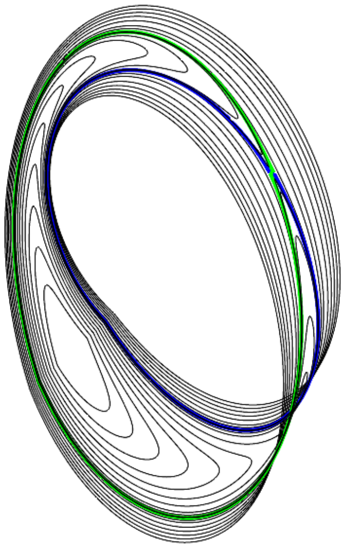

Figure 5: Quadratic reversible double centers in (7) compactified to the Moebius strip $\mathbb{RP}^2 \setminus \mathbb{D}$. One of the two centers is depicted at the front of the drawing, while we placed the other one in the removed invariant disk $\mathbb{D}$ for convenience. The polycycle $\bar{\Gamma}_u = \bar{\Gamma}_\ell$ is represented by the two circles in blue and green intersecting at the saddle point at the back.

Hausdorff sense are in one to one correspondence with the isolated positive zeroes of the *difference map*

$$\mathscr{D}(s;\mu) := D_+(s;\mu) - D_-(s;\mu)$$

near $s = 0$. The following result gives the asymptotic development of $\mathscr{D}(s;\mu)$ at $s = 0$ and the functions λ, F_1, F_2, F_3 and d_0 in its statement are the ones defined in (2), (3), (4), (5) and (6), respectively. In the statement we use the Ecalle-Roussarie comensator $\omega(s;\alpha)$, see Definition A.1, and $\mathcal{F}_\ell^\infty(\mu_0)$ stands for a function ℓ-flat with respect to s at μ_0, see Definition A.3.

Theorem 2.1. *Let us fix any $\mu_0 \in \Lambda$ and set $\lambda_0 := \lambda(\mu_0)$. Then $\mathscr{D}(s;\mu) = \Delta_0(\mu)s^\lambda + \mathcal{F}_\ell^\infty(\mu_0)$ for any $\ell \in \big[\lambda_0, \min(2\lambda_0, \lambda_0+1)\big)$, where Δ_0 is an analytic function at μ_0 that can be written as $\Delta_0 = \kappa_0 d_0$, with κ_0 analytic at μ_0 and $\kappa_0(\mu_0) > 0$. In addition,*

(1) *If $\lambda_0 > 1$ then $\mathscr{D}(s;\mu) = \Delta_0(\mu)s^\lambda + \Delta_1(\mu)s^{\lambda+1} + \mathcal{F}_\ell^\infty(\mu_0)$ for any $\ell \in \big[\lambda_0+1, \min(2\lambda_0, \lambda_0+2)\big)$. Furthermore Δ_1 is an analytic function at μ_0 that can be written as $\Delta_1 = \kappa_1 F_1 + \bar{\kappa}_1\Delta_0$, where κ_1 and $\bar{\kappa}_1$ are analytic at μ_0 and $\kappa_1(\mu_0) > 0$.*

(2) *If $\lambda_0 < 1$ then $\mathscr{D}(s;\mu) = \Delta_0(\mu)s^\lambda + \Delta_2(\mu)s^{2\lambda} + \mathcal{F}_\ell^\infty(\mu_0)$ for any $\ell \in \big[2\lambda_0, \min(3\lambda_0, \lambda_0+1)\big)$. Moreover Δ_2 is an analytic function at μ_0 that can be written as $\Delta_2 = \kappa_2 F_2 + \bar{\kappa}_2\Delta_0$, where κ_2 and $\bar{\kappa}_4$ are analytic at μ_0 and $\kappa_2(\mu_0) > 0$.*

(3) *If $\lambda_0 = 1$ then $\mathscr{D}(s;\mu) = \Delta_0(\mu)s^\lambda + \Delta_3(\mu)s^{\lambda+1}\omega(s;1-\lambda) + \Delta_4(\mu)s^{\lambda+1} + \mathcal{F}_\ell^\infty(\mu_0)$ for any $\ell \in [2,3)$ and*

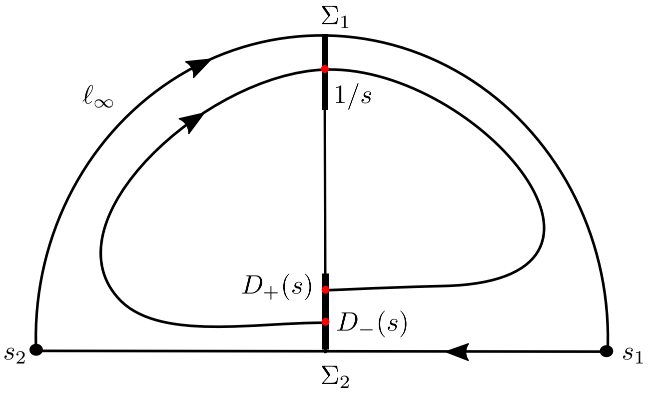

Figure 6: Dulac maps for the definition of $\mathscr{D} = D_+ - D_-$ in Theorem 2.1

where Δ_3 *and* Δ_4 *are analytic functions at* μ_0*. Moreover there exist analytic functions* κ_3 *and* $\bar{\kappa}_3$ *at* μ_0 *with* $\kappa_3(\mu_0) > 0$ *such that the equality* $\Delta_3 = \kappa_3 F_3 + \bar{\kappa}_3 \Delta_0$ *holds on* $\{\mu \in \Lambda : \lambda(\mu) = 1\}$*.*

Since the proof of Theorem 2.1 is rather long and technical and also requires several results from previous papers, we postpone it to Appendix B for reader's convenience.

Remark 2.2. On account of the definition of d_1 given in (6), Theorem 2.1 provides the following information about the stability of the polycycle Γ_u for the vector field X_{μ_0}:

(*a*) If $d_0(\mu_0) < 0$ (respectively, $d_0(\mu_0) > 0$) then Γ_u is asymptotically stable (respectively, unstable).

(*b*) If $d_0(\mu_0) = 0$ and $d_1(\mu_0) < 0$ (respectively, $d_1(\mu_0) > 0$) then Γ_u is asymptotically stable (respectively, unstable).

The key point for this observation is that the functions κ_i in the statement of Theorem 2.1 are strictly positive at μ_0. □

For simplicity in the exposition, from now on we will use the following definition.

Definition 2.3. Let $h(s;\mu)$ be a function in $\mathscr{C}^\infty_{s>0}(U)$ for some open set $U \subset \mathbb{R}^N$. Given any $\mu_0 \in U$ we define $\mathcal{Z}_0(h(\,\cdot\,;\mu),\mu_0)$ to be the smallest integer ℓ having the property that there exist $\delta > 0$ and a neighbourhood V of μ_0 such that for every $\mu \in V$ the function $h(s;\mu)$ has no more than ℓ isolated zeros on $(0,\delta)$ counted with multiplicities. □

Proof of Theorem A. Recall (see Figure 6) that the limit cycles of the vector field (1) that are close to Γ_u in Hausdorff sense are in one to one correspondence with the isolated positive zeroes of the difference map

$$\mathscr{D}(s;\mu) = D_+(s;\mu) - D_-(s;\mu)$$

near $s = 0$. Hence, see Definition 2.3, we have that $\mathrm{Cycl}\big((\Gamma_u, X_{\mu_0}), X_\mu\big) \leqslant \mathcal{Z}_0\big(\mathscr{D}(\,\cdot\,;\mu),\mu_0\big)$. Note moreover that, by Theorem 2.1,

$$\mathscr{D}(s;\mu) = \Delta_0(\mu)s^\lambda + \mathcal{F}^\infty_\ell(\mu_0) \tag{9}$$

for any $\ell \in \big[\lambda_0, \min(2\lambda_0, \lambda_0+1)\big)$, where $\lambda_0 := \lambda(\mu_0)$ and $\Delta_0 = \kappa_0 d_0$ with $\kappa_0(\mu_0) > 0$. If $d_0(\mu_0) \neq 0$ then, taking any $\ell > \lambda_0$ (see Definition A.3),

$$\lim_{(s,\mu)\to(0^+,\mu_0)} s^{-\lambda}\mathscr{D}(s;\mu) = \Delta_0(\mu_0) \neq 0,$$

which implies $\mathcal{Z}_0\big(\mathscr{D}(\,\cdot\,;\mu),\mu_0\big)=0$ and proves (a).

On the other hand, since $\mathscr{D}(\,\cdot\,;\mu_0)\equiv 0$ if, and only if, $\mathscr{R}_u(\,\cdot\,;\mu_0)\equiv \mathrm{Id}$, the assertion in (b) follows from the equality in (9) by applying Proposition A.12 with $n=1$.

We turn next to the proof of (c) and (d). To this end we shall use that, by applying Theorem 2.1,

$$\mathscr{D}(s;\mu)=\Delta_0(\mu)s^{\lambda}+\begin{cases}\Delta_1(\mu)s^{\lambda+1}+\mathcal{F}^{\infty}_{\ell_1}(\mu_0) & \text{if } \lambda_0>1,\\ \Delta_2(\mu)s^{2\lambda}+\mathcal{F}^{\infty}_{\ell_2}(\mu_0) & \text{if } \lambda_0<1,\\ \Delta_3(\mu)s^{\lambda+1}\omega(s;1-\lambda)+\Delta_4(\mu)s^{\lambda+1}+\mathcal{F}^{\infty}_{\ell_3}(\mu_0) & \text{if } \lambda_0=1,\end{cases} \tag{10}$$

for any $\ell_1\in\big[\lambda_0+1,\min(2\lambda_0,\lambda_0+2)\big)$, $\ell_2\in\big[2\lambda_0,\min(3\lambda_0,\lambda_0+1)\big)$ and $\ell_3\in[2,3)$, respectively. Moreover, in its respective case, the coefficient Δ_i is an analytic function at μ_0. In addition, for $i\in\{0,1,2,3\}$, there exist analytic functions κ_i and $\bar{\kappa}_i$ at μ_0 with $\kappa_i(\mu_0)>0$ such that we can write

$$\Delta_0=\kappa_0 d_0,\quad \Delta_1=\kappa_1F_1+\bar{\kappa}_1\Delta_0,\quad \Delta_2=\kappa_2F_2+\bar{\kappa}_2\Delta_0 \text{ and } \Delta_3|_{\Lambda_1}=(\kappa_3F_3+\bar{\kappa}_3\Delta_0)|_{\Lambda_1} \tag{11}$$

where recall that $\Lambda_1:=\{\mu\in\Lambda:\lambda(\mu)=1\}$.

In order to show (c) we can suppose that $\Delta_0=\kappa_0d_0$ vanishes at μ_0 because otherwise we have already proved that $\mathrm{Cycl}\big((\Gamma_u,X_{\mu_0}),X_\mu\big)=0$. On account of this the assumption $d_1(\mu_0)\neq 0$ implies, see the definition given in (6), that $\Delta_1(\mu_0)\neq 0$ if $\lambda_0>1$, $\Delta_2(\mu_0)\neq 0$ if $\lambda_0<1$ and $\Delta_3(\mu_0)\neq 0$ if $\lambda_0=1$. In the first case, from (10) and applying Lemma A.7,

$$\begin{aligned}\partial_s\big(s^{-\lambda}\mathscr{D}(s;\mu)\big)&=\partial_s\big(\Delta_0(\mu)+\Delta_1(\mu)s+s^{-\lambda}\mathcal{F}^{\infty}_{\ell_1}(\mu_0)\big)\\&=\Delta_1(\mu)-\lambda s^{-\lambda-1}\mathcal{F}^{\infty}_{\ell_1}(\mu_0)+s^{-\lambda}\mathcal{F}^{\infty}_{\ell_1-1}(\mu_0)\\&=\Delta_1(\mu)+\mathcal{F}^{\infty}_{\varepsilon}(\mu_0)\end{aligned}$$

for some $\varepsilon>0$ small enough since we can take $\ell_1>\lambda_0+1$. Therefore, see Definition A.3, the derivative $\partial_s\big(s^{-\lambda}\mathscr{D}(s;\mu)\big)$ tends to $\Delta_1(\mu_0)\neq 0$ as $(s,\mu)\to(0^+,\mu_0)$. Thus, by applying Rolle's Theorem,

$$\mathrm{Cycl}\big((\Gamma_u,X_{\mu_0}),X_\mu\big)\leqslant \mathcal{Z}_0\big(\mathscr{D}(\,\cdot\,;\mu),\mu_0\big)\leqslant 1,$$

as desired. Similarly, in the second case (i.e., $\lambda_0<1$) we have that

$$\partial_s\big(s^{-\lambda}\mathscr{D}(s;\mu)\big)=\lambda\Delta_2(\mu)s^{\lambda-1}+\mathcal{F}^{\infty}_{\varepsilon}(\mu_0)=s^{\lambda-1}\big(\lambda\Delta_2(\mu)+s^{1-\lambda}\mathcal{F}^{\infty}_{\varepsilon}(\mu_0)\big)$$

for some $\varepsilon>0$ small enough. Then, due to $\Delta_2(\mu_0)\neq 0$, we conclude by Rolle's Theorem as before that $\mathcal{Z}_0\big(\mathscr{D}(\,\cdot\,;\mu),\mu_0\big)\leqslant 1$. If $\lambda_0=1$ then, from (10) once again and taking $\ell_3\in[2,3)$ into account, the application of Lemma A.7 yields

$$\begin{aligned}\partial_s\big(s^{-\lambda}\mathscr{D}(s;\mu)\big)&=\partial_s\big(\Delta_0(\mu)+\Delta_3(\mu)s\omega(s;1-\lambda)+\Delta_4(\mu)s+s^{-\lambda}\mathcal{F}^{\infty}_{\ell_3}(\mu_0)\big)\\&=\Delta_3(\mu)\big(\lambda\omega(s;1-\lambda)-1\big)+\Delta_4(\mu)+\mathcal{F}^{\infty}_{\varepsilon}(\mu_0)\end{aligned}$$

for $\varepsilon>0$ small enough. Here we use that $\partial_s s\omega(s;\alpha)=(1-\alpha)\omega(s;\alpha)-1$, see Definition A.1. Consequently, if $(s,\mu)\to(0^+,\mu_0)$ then

$$\frac{\partial_s\big(s^{-\lambda}\mathscr{D}(s;\mu)\big)}{\omega(s;1-\lambda)}=\Delta_3(\mu)\left(\lambda-\frac{1}{\omega(s;1-\lambda)}\right)+\frac{\Delta_4(\mu)}{\omega(s;1-\lambda)}+\frac{\mathcal{F}^{\infty}_{\varepsilon}(\mu_0)}{\omega(s;1-\lambda)}\to\lambda_0\Delta_3(\mu_0)\neq 0,$$

since $\lim_{(s,\alpha)\to(0^+,0)}\frac{1}{\omega(s;\alpha)}=0$ by (a) in [16, Lemma A.4]. By Rolle's Theorem again, this implies that $\mathcal{Z}_0\big(\mathscr{D}(\,\cdot\,;\mu),\mu_0\big)\leqslant 1$ in the case $\lambda_0=1$ as well and completes the proof of assertion (c).

Let us show finally the validity of the two assertions in (d). The first one concerns the case $\mu_0 \notin \Lambda_1$, i.e., $\lambda_0 \neq 1$. If $\lambda_0 > 1$ then, from (10),

$$\begin{aligned} s^{-\lambda}\mathscr{D}(s;\mu) &= \Delta_0(\mu) + \Delta_1(\mu)s + f_2(s;\mu) \\ &= \kappa_0 d_0 + (\kappa_1 F_1 + \bar{\kappa}_1\kappa_0 d_0)s + f_2(s;\mu) \\ &= d_0\kappa_0(1+\bar{\kappa}_1 s) + d_1\kappa_1 s + f_2(s;\mu), \end{aligned}$$

where in the first equality $f_2 \in s^{-\lambda}\mathcal{F}^\infty_{\ell_1}(\mu_0) \subset \mathcal{F}^\infty_{1+\varepsilon}(\mu_0)$ for $\varepsilon > 0$ small enough by Lemma A.7 due to $\ell_1 > \lambda_0 + 1$, in the second one we take (11) into account, and in the third one that $d_1(\mu) = F_1(\mu)$ if $\lambda(\mu) > 1$. Thus, setting $f_0(s;\mu) = \kappa_0(1+\bar{\kappa}_1 s)$ and $f_1(s;\mu) = \kappa_1 s$, we can write

$$s^{-\lambda}\mathscr{D}(s;\mu) = d_0(\mu)f_0(s;\mu) + d_1(\mu)f_1(s;\mu) + f_2(s;\mu). \tag{12}$$

By assumption we have that d_0 and d_1 vanish and are independent at μ_0 and that $\mathscr{D}(\,\cdot\,;\mu_0) \not\equiv 0$ due to $\mathscr{R}_u(\,\cdot\,;\mu_0) \not\equiv \mathrm{Id}$. Accordingly, since $\frac{f_1(s;\mu)}{f_0(s;\mu)} = \frac{\kappa_1 s}{\kappa_0(1+\bar{\kappa}_1 s)}$ and $\frac{f_2(s;\mu)}{f_1(s;\mu)} \in s^{-1}\mathcal{F}^\infty_{1+\varepsilon}(\mu_0)$ tend to zero as $s \to 0^+$, we can apply Proposition A.12 with $n = 2$ to conclude that $\mathrm{Cycl}\big((\Gamma_u, X_{\mu_0}), X_\mu\big) \geqslant 2$. If $\lambda_0 < 1$ then following verbatim from (10) and (11) we get the equality in (12) with $f_0(s;\mu) = \kappa_0(1+\bar{\kappa}_2 s^\lambda)$, $f_1(s;\mu) = \kappa_2 s^\lambda$ and $f_2 \in s^{-\lambda}\mathcal{F}^\infty_{\ell_2}(\mu_0) \subset \mathcal{F}^\infty_{\lambda_0+\varepsilon}(\mu_0)$. Thus the assumptions in Proposition A.12 are also verified and so the lower bound $\mathrm{Cycl}\big((\Gamma_u, X_{\mu_0}), X_\mu\big) \geqslant 2$ is true for the case $\lambda_0 > 1$ as well. Let us consider finally the case $\lambda_0 = 1$, which is slightly different. In this case, from (10) and taking Definition A.1 into account, if $\mu \in \Lambda_1$ then

$$\begin{aligned} s^{-\lambda}\mathscr{D}(s;\mu) &= \Delta_0(\mu) - \Delta_3(\mu)s\log s + \Delta_4(\mu)s + \hat{f}_2(s;\mu) \\ &= d_0\kappa_0(1-\bar{\kappa}_3 s\log s) - F_3\kappa_3 s\log s + \Delta_4 s + \hat{f}_2(s;\mu), \end{aligned}$$

where in the first equality $\hat{f}_2 \in s^{-1}\mathcal{F}^\infty_{\ell_3}(\mu_0) \subset \mathcal{F}^\infty_{1+\varepsilon}(\mu_0)$ and the second one follows from (11) due to $\mu \in \Lambda_1$. Hence, since $d_1 = F_3$ on Λ_1, we can write

$$s^{-\lambda}\mathscr{D}(s;\mu)\big|_{\mu\in\Lambda_1} = d_0|_{\Lambda_1}\, f_0(s;\mu) + d_1|_{\Lambda_1}\, f_1(s;\mu) + f_2(s;\mu)$$

taking the functions $f_0(s;\mu) = \kappa_0(1-\bar{\kappa}_3 s\log s)$, $f_1(s;\mu) = -\kappa_3 s\log s$ and $f_2(s;\mu) = \Delta_4 s + \hat{f}(s;\mu)$. Once again, $\frac{f_1(s;\mu)}{f_0(s;\mu)} = -\frac{\kappa_3 s\log s}{\kappa_0(1-\bar{\kappa}_3 s\log s)}$ and $\frac{f_2(s;\mu)}{f_1(s;\mu)} = -\frac{\Delta_4 + s^{-1}\hat{f}(s;\mu)}{\kappa_3\log s}$ tend to zero as $s \to 0^+$ and, on the other hand, $d_1|_{\Lambda_1}$ and $d_1|_{\Lambda_1}$ vanish and are independent at μ_0 by assumption. Consequently, by applying Proposition A.12 with $W = \Lambda_1$ and $n = 2$ we get that $\mathrm{Cycl}\big((\Gamma_u, X_{\mu_0}), X_\mu\big) \geqslant 2$ in case that $\lambda_0 = 1$, as desired. This proves the second assertion in (d) and concludes the proof of the result. ■

3 Proof of Theorem B

The following result shows that to prove Theorem B it suffices to consider a 5-dimensional perturbation.

Lemma 3.1. *Any quadratic differential system which is close (in the topology of coefficients) to (7) for some $(a_0, b_0) \in \mathbb{R}^2$ with $a_0 \neq -2$ can be brought by means of an affine change of coordinates and a constant rescaling of time to*

$$X_\mu \quad \begin{cases} \dot{x} = \frac{b-2}{4} + \varepsilon_1 x + (1-b)y + ax^2 + \varepsilon_2 xy + by^2, \\ \dot{y} = \varepsilon_0 - 2xy, \end{cases} \tag{13}$$

with $(a, b, \varepsilon_0, \varepsilon_1, \varepsilon_2) \approx (a_0, b_0, 0, 0, 0)$.

Proof. We consider the group $\mathrm{Aff}(2,\mathbb{R})$ of affine transformations

$$g(x,y) = (g_{11}x + g_{12}y + g_{13}, g_{21}x + g_{22}y + g_{23})$$

and the pull-back $g^\star(Y_{a,b}) = \big(Dg^{-1}\big)(Y_{a,b}\circ g)$ of

$$Y_{a,b} := \left((1-b)y + by^2 + \frac{b-2}{4} + ax^2\right)\partial_x - 2xy\partial_y.$$

Note that $Y_{a,b} = w_0 + aw_1 + bw_2$ with $w_0 := (y-\frac{1}{2})\partial_x - 2xy\partial_y$, $w_1 := x^2\partial_x$ and $w_2 := (-y+y^2+\frac{1}{4})\partial_x$. An easy computation performed with `Maple` shows that if $a_0 \neq -2$ then the vector fields $v_0 = \partial_y$, $v_1 = x\partial_x$ and $v_2 = xy\partial_x$ span a complementary to the tangent space at the point $(\lambda, g, a, b) = (1, \mathrm{id}, a_0, b_0)$ of the orbit

$$\{\lambda g^\star(Y_{a,b}) : \ \lambda \in \mathbb{R}^*, \ g \in \mathrm{Aff}, \ a, b \in \mathbb{R}\}$$

in the 12-dimensional space $\mathcal{P}_2$ of all polynomial vector fields of degree 2. In other words, if $a_0 \neq -2$ then the map $F : U := \mathbb{R}^* \times \mathrm{Aff}(2,\mathbb{R}) \times \mathbb{R}^5 \to \mathcal{P}_2$ defined by

$$F(\lambda, g, a, b, \varepsilon_0, \varepsilon_1, \varepsilon_2) = \lambda g^\star Y_{a,b} + \varepsilon_0 v_0 + \varepsilon_1 v_1 + \varepsilon_2 v_2$$

is a local diffeomorphism between neighbourhoods of $(1, \mathrm{id}, a_0, b_0, 0, 0, 0)$ in U and Y_{a_0,b_0} in $\mathcal{P}_2$. This proves the result. ∎

We stress that henceforth X_μ refers to the differential system in (13). That being said, the key point for our purposes is that X_μ writes as

$$\begin{cases} \dot{x} = yf(x,y;\mu) + g(x;\mu), \\ \dot{y} = \varepsilon_0 + yq(x,y;\mu), \end{cases}$$

with $f(x,y) = (1-b) + \varepsilon_2 x + by$, $g(x) = \frac{b-2}{4} + \varepsilon_1 x + ax^2$ and $q(x,y) = -2x$, so that X_μ is a D-system for $\varepsilon_0 = 0$. Moreover one can easily check that X_μ with $a \in (-2,0)$, $b \in (0,2)$, $\varepsilon_0 = 0$, $\varepsilon_1 \approx 0$ and $\varepsilon_2 \approx 0$ verifies assumptions **H1** and **H2**. Accordingly, for these parameter values, X_μ has a polycycle Γ_u at the boundary of the upper half-plane with two hyperbolic saddles, $s_1 = \{y=0, x>0\} \cap \ell_\infty$ and $s_2 = \{y=0, x<0\} \cap \ell_\infty$. Since ε_0 does not affect the homogenous part of higher degree of X_μ, the location and character of these two singular points remains unaltered taking $\varepsilon_i \approx 0$ for $i = 0,1,2$.

Let us fix any $\mu_0 = (a_0, b_0, \varepsilon_0, \varepsilon_1, \varepsilon_2)$ with $(a_0, b_0) \in (-2,0) \times (0,2)$ and $\varepsilon_i \approx 0$ for $i = 0,1,2$. We take two transverse sections on $x = 0$: Σ_1, parametrized by $s \mapsto (0, 1/s)$ with $s \in (0,\delta)$, and Σ_2, parametrized by $s \mapsto (0,s)$ with $s \in (-\delta,\delta)$. By continuity with respect to initial conditions and parameters, for $\mu \approx \mu_0$ and $\delta > 0$ small enough, we have a well defined Dulac map $D_+^u(\,\cdot\,;\mu)$ for X_μ from Σ_1 to Σ_2 and a well defined Dulac map $D_-^u(\,\cdot\,;\mu)$ for $-X_\mu$ from Σ_1 to Σ_2, see Figure 7. In our next result we study the asymptotic development of the difference map

$$\mathscr{D}_u(s;\mu) := D_+^u(s;\mu) - D_-^u(s;\mu).$$

It is clear that the positive zeros of this function are in one-to-one correspondence with the limit cycles of X_μ bifurcating from Γ_u to the upper half-plane.

Proposition 3.2. *Fix any $\mu_0 = (a_0, b_0, 0, 0, 0)$ with $(a_0, b_0) \in (-2,0) \times (0,2)$. Then*

$$\mathscr{D}_u(s;\mu) = \delta_u + \Delta_0^u s^\lambda + \mathcal{F}_L^\infty(\mu_0), \text{ for any } L \in \big[\lambda_0, \min(2\lambda_0, \lambda_0+1)\big),$$

where λ, δ_u and Δ_0^u are smooth functions in a neighbourhood of μ_0 and $\lambda_0 := \lambda(\mu_0) = -\frac{a_0+2}{a_0}$. In addition $\mathscr{D}_u(s;\mu_0) \equiv 0$, $\partial_{\varepsilon_0}\delta_u(\mu_0) < 0$, $\partial_{\varepsilon_1}\delta_u(\mu_0) = \partial_{\varepsilon_2}\delta_u(\mu_0) = 0$ and

$$\Delta_0^u(\mu) = -\kappa_{01}(\mu)\left(2\frac{\sqrt{b(a+2)}}{\sqrt{a(b-2)}}\,\varepsilon_1 + \varepsilon_2\right) + \kappa_{02}(\mu)\delta_u(\mu),$$

where κ_{0i} are smooth functions at $\mu = \mu_0$ for $i = 1,2$ and $\kappa_{01}(\mu_0) > 0$. Furthermore the following assertions are also true in case that $a_0 \neq -1$:

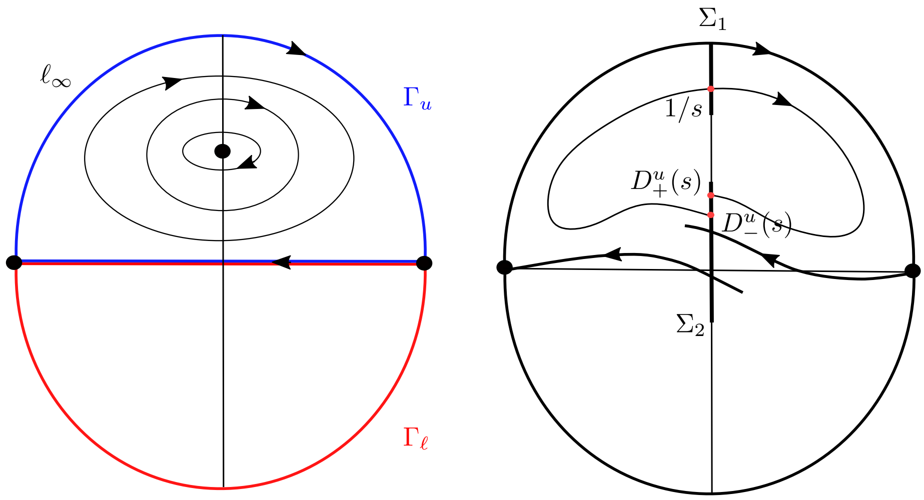

Figure 7: Phase portrait in the Poincaré disc of the vector field X_μ in (13) for $\varepsilon_0=\varepsilon_1=\varepsilon_2=0$ (left) and $\varepsilon_0\neq 0$ (right). On the right, Dulac maps $D_\pm$ to define the function $\mathscr{D}_u(s;\mu)=D_+^u(s;\mu)-D_-^u(s;\mu)$ studied in Proposition 3.2. The points in red are $(0,D_\pm^u(s))$ and $(0,1/s)$.

(1) *If $a_0>-1$ then $\mathscr{D}_u(s;\mu)=\delta+\Delta_0^u s^\lambda+\Delta_1^u s^{\lambda+1}+\mathcal{F}_L^\infty(\mu_0)$ for any $L\in\big[\lambda_0+1,\min(2\lambda_0,\lambda_0+2)\big)$, where Δ_1^u is a smooth function in a neighbourhood of μ_0 satisfying that*

$$\Delta_1^u(\mu)=\kappa_{11}(\mu)\left(\varepsilon_1+\tfrac{a(b-1)}{2(a+1)b}\varepsilon_2+\mathrm{o}(\|(\varepsilon_1,\varepsilon_2)\|)\right)+\kappa_{12}(\mu)\Delta_0^u(\mu)+\kappa_{13}(\mu)\delta_u(\mu)$$

where κ_{1i} are smooth functions at $\mu=\mu_0$ for $i=1,2,3$ and $\kappa_{11}(\mu_0)>0$.

(2) *If $a_0<-1$ then $\mathscr{D}_u(s;\mu)=\delta_u+\Delta_0^u s^\lambda+\Delta_2^u s^{2\lambda}+\mathcal{F}_L^\infty(\mu_0)$ for any $L\in\big[2\lambda_0,\min(3\lambda_0,\lambda_0+1)\big)$, where Δ_2^u is a smooth function in a neighbourhood of μ_0 satisfying that*

$$\Delta_2^u(\mu)=\kappa_{21}(\mu)\left(\tfrac{2(a+2)(b-1)}{(a+1)(b-2)}\varepsilon_1+\varepsilon_2+\mathrm{o}(\|(\varepsilon_1,\varepsilon_2)\|)\right)+\kappa_{22}(\mu)\Delta_0^u(\mu)+\kappa_{23}(\mu)\delta_u(\mu),$$

where κ_{2i} are smooth functions at $\mu=\mu_0$ for $i=1,2,3$ and $\kappa_{21}(\mu_0)>0$.

For reader's convenience the proof of Proposition 3.2 is deferred to Subsection B.2.

Let us fix $\mu_0=(a_0,b_0,0,0,0)$ with $a_0\in(-2,0)$ and $b_0\in(0,2)$. The differential system (13) has only two finite singularities for $\mu\approx\mu_0$, which are of focus type and close to the points $(0,\frac{1}{2})$ and $(0,\frac{b_0-2}{2b_0})$. Let us denote them by $c_u(\mu)$ and $c_\ell(\mu)$, respectively. We also define the parameter subset

$$\mathscr{Z}_u:=\{\mu\approx\mu_0:\mathscr{D}_u(\,\cdot\,;\mu)\equiv 0\}.$$

The next result shows that $\mathscr{Z}_u$ is precisely the center manifold for the focus at $(0,\frac{1}{2})$. We remark, in connection with our discussion in Figure 4, that the subsets Z_0 and Z_1 correspond to the components Q_3^R and Q_3^{LV}, respectively. For completeness, we note that the combination of this result with Lemma 4.1 provides also the description of the center manifold for the focus at $(0,\frac{b_0-2}{2b_0})$.

Lemma 3.3. *$\mathscr{Z}_u=\{\mu\approx\mu_0:c_u(\mu)$ is a center of $X_\mu\}$ and $\mathscr{Z}_u=Z_0\cup Z_1$, where*

$$Z_0:=\{\mu\approx\mu_0:\varepsilon_0=\varepsilon_1=\varepsilon_2=0\}\text{ and }Z_1:=\{\mu\approx\mu_0:a+b=\varepsilon_0=2\varepsilon_1+\varepsilon_2=0\}.$$

Moreover, if $\mu\in\mathscr{Z}_u$ then the period annulus of the center at $(0,\frac{1}{2})$ is $\big\{(x,y)\in\mathbb{R}^2:y>0\big\}\setminus\{(0,\frac{1}{2})\}$.

Proof. Let us fix $\hat{\mu} \approx \mu_0$ and consider the straight line L passing through the singularities $c_u(\hat{\mu})$ and $c_\ell(\hat{\mu})$. These two points split L into three open segments where $X_{\hat{\mu}}$ is transverse because the vector field is quadratic. Let us denote the unbounded segment having $c_u(\hat{\mu})$ as endpoint by Σ_1 and the bounded segment by Σ_2. We parametrize them analytically by $\sigma_1 : (0,1) \longrightarrow \Sigma_1$ and $\sigma_2 : (0,1) \longrightarrow \Sigma_2$, respectively, such that $\lim_{s\to 0} \|\sigma_1(s)\| = +\infty$, $\lim_{s\to 1} \sigma_1(s) = c_u(\hat{\mu})$, $\lim_{s\to 0} \sigma_2(s) = c_\ell(\hat{\mu})$ and $\lim_{s\to 1} \sigma_2(s) = c_u(\hat{\mu})$. By transversality and the fact that $c_u(\hat{\mu})$ and $c_\ell(\hat{\mu})$ are the only finite singularities of $X_{\hat{\mu}}$, the application of the Poincaré-Bendixson Theorem shows that there is a well defined Poincaré map for $X_{\hat{\mu}}$ from Σ_1 to Σ_2. Taking the parametrizations previously introduced, we denote it by $\mathcal{P}_+ : (0,1) \longrightarrow (0,1)$, which is an analytic function by applying the Implicit Function Theorem. Similarly, we denote by $\mathcal{P}_- : (0,1) \longrightarrow (0,1)$ the Poincaré map for $-X_{\hat{\mu}}$ from Σ_1 to Σ_2, which is analytic as well. Observe that, by construction, the periodic orbits surrounding $c_u(\hat{\mu})$ correspond to zeros of $\mathcal{D} := \mathcal{P}_+ - \mathcal{P}_-$. Moreover $\hat{\mu} \in \mathcal{Z}_u$ if, and only if, $\mathcal{D} \equiv 0$ on $(0, \delta_1)$ and, on the other hand, $c_u(\hat{\mu})$ is a center if, and only if, $\mathcal{D} \equiv 0$ on $(1-\delta_2, 1)$. Accordingly, since $\mathcal{D}$ is analytic on $(0,1)$, this proves that $\hat{\mu} \in \mathcal{Z}_u$ if, and only if, $c_u(\hat{\mu})$ is a center. So far we have proved that

$$\mathcal{Z}_u = \{\mu \approx \mu_0 : c_u(\mu) \text{ is a center of } X_\mu\} =: U.$$

Our next task is to show that $U = Z_0 \cup Z_1$. To prove the inclusion $U \subset Z_0 \cup Z_1$ we take any $\mu \in U$ and, due to $U = \mathcal{Z}_u$, by applying Proposition 3.2 we get that $\delta_u(\mu) = 0$ and $\Delta_0^u(\mu) = 0$, which imply

$$\varepsilon_0 = 0 \text{ and } 2\frac{\sqrt{b(a+2)}}{\sqrt{a(b-2)}}\,\varepsilon_1 + \varepsilon_2 = 0.$$

Here the first equality follows by the Implicit Function Theorem using that $\delta_u|_{\varepsilon_0} \equiv 0$ and $\partial_{\varepsilon_0}\delta_u(\mu_0) \neq 0$. Recall on the other hand that trace equal to zero is a necessary condition for a singular point to be a center. One can verify that if $\varepsilon_0 = 0$ then $c_u(\mu) = (0, \frac{1}{2})$ and that its trace is equal to $\varepsilon_1 + \frac{1}{2}\varepsilon_2$. The vanishing of this quantity, together with the two equalities above, yields to either $\{\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 0\}$ or $\{a + b = \varepsilon_0 = 2\varepsilon_1 + \varepsilon_2 = 0\}$. Therefore $U \subset Z_0 \cup Z_1$. To prove the reverse inclusion we note first that if $\mu \in Z_0$ then the function

$$H_0(x,y) = |y|^a(x^2 + ly^2 + my + n),$$

with $l = \frac{b}{a+2}$, $m = -\frac{b-1}{a+1}$ and $n = \frac{b-2}{4a}$, is a global first integral of X_μ. The continuity of H_0 at $c_u(\mu)$ implies that it must be a center, so that $\mu \in U$. Finally, if $\mu \in Z_1$ then one can verify that

$$\begin{aligned} H_1(x,y) &= |y|^a(r_1(x,y) + i\alpha_1 x)^{1-i\frac{\varepsilon_2}{\alpha_1}}(r_1(x,y) - i\alpha_1 x)^{1+i\frac{\varepsilon_2}{\alpha_1}} \\ &= |y|^a(r_1(x,y)^2 + \alpha_1^2 x^2)e^{\frac{2\varepsilon_2}{\alpha_1}\arg(r_1(x,y)+i\alpha_1 x)}, \end{aligned}$$

with $r_1(x,y) = 2by + (2-b) + \varepsilon_2 x$ and $\alpha_1 = \sqrt{4b(2-b) - \varepsilon_2^2}$, is a well defined first integral of X_μ outside any ray from $\{r_1(x,y) = 0, x = 0\} = \{c_\ell(\mu)\}$ to infinity. In particular it is continuous at $c_u(\mu)$, so that again it must be a center and $\mu \in U$. This proves the result. ■

Lemma 3.4. *Suppose that $F(u_1,u_2,v)$ is a smooth function on a neighbourhood U of $(0,0,v_0) \in \mathbb{R}^2 \times \mathbb{R}^n$ verifying $F = \mathrm{o}(\|(u_1,u_2)\|)$. Then there exist smooth functions $F_1(u_1,u_2,v)$ and $F_2(u_2,v)$ on U such that $F(u_1,u_2,v) = u_1F_1(u_1,u_2,v) + u_2^2F_2(u_2,v)$.*

Proof. The hypothesis implies that $F(0,0,v) \equiv 0$ and $\partial_{u_i}F(0,0,v) \equiv 0$. Then

$$\begin{aligned} F(u_1,u_2,v) &= F(u_1,u_2,v) - F(0,u_2,v) + F(0,u_2,v) - F(0,0,v) \\ &= u_1\underbrace{\int_0^1 \partial_{u_1}F(tu_1,u_2,v)dt}_{F_1(u_1,u_2,v)} + u_2\underbrace{\int_0^1 \partial_{u_2}F(0,tu_2,v)dt}_{G(u_2,v)} \end{aligned}$$

where F_1 and G are smooth functions on U. Since $G(0,v) = \partial_{u_2}F(0,0,v) = 0$, we also deduce that $G(u_2,v) = u_2F_2(u_2,v)$ where

$$F_2(u_2,v) = \int_0^1 \partial_{u_2}G(tu_2,v)dt$$

is also smooth on U. Hence we can write $F = u_1F_1 + u_2^2F_2$ and the result follows. ■

In the statement of our next result, and in what follows, we denote

$$\varepsilon_\pm(\mu) = \varepsilon_2 \pm 2\sqrt{\frac{b(a+2)}{a(b-2)}}\varepsilon_1 \text{ and } c_\pm(\mu) = (a+1) \pm (1-b). \tag{14}$$

Theorem 3.5. *Given any $\mu_0 = (a_0, b_0, 0, 0, 0)$ with $a_0 \in (-2,0) \setminus \{-1\}$ and $b_0 \in (0,2)$, there exist a neighbourhood U of μ_0 in $\mathbb{R}^5$ and $\delta > 0$ such $\nu = \Phi(\mu) := (\varepsilon_0, \varepsilon_+, \varepsilon_-, c_+, c_-)$ is a local change of coordinates in U and we can write*

$$\mathscr{D}_u(s;\mu)|_{\mu=\Phi^{-1}(\nu)} = -\nu_1 g_1(s;\nu) - \nu_2 g_2(s;\nu) - \nu_3\nu_5 g_3(s;\nu), \tag{15}$$

where, setting $\nu_0 = \Phi(\mu_0) = (0,0,0,\nu_4^0,\nu_5^0)$,

(a) $g_1(s;\nu) = \kappa_1(\nu) + \mathcal{F}_\delta^\infty(\nu_0)$,

(b) $g_2(s;\nu) = s^{\underline{\lambda}(\nu)}\big(\kappa_2(\nu) + \mathcal{F}_\delta^\infty(\nu_0)\big)$ *where* $\underline{\lambda}(\nu)|_{\nu=\Phi(\mu)} = -\frac{a+2}{a}$, *and*

(c) $g_3(s;\nu) = s^{\underline{\lambda}'(\nu)}\big(\kappa_3(\nu) + \mathcal{F}_\delta^\infty(\nu_0)\big)$ *where* $\underline{\lambda}'(\nu) = \underline{\lambda}(\nu) + \min\big(\underline{\lambda}(\nu), 1\big)$.

Moreover κ_1, κ_2 and κ_3 are smooth strictly positive functions on $\Phi(U)$.

Proof. The result is a consequence of Proposition 3.2. Note first that, since $\partial_{\varepsilon_0}\delta_u(\mu_0) < 0$ and $\delta_u|_{\varepsilon_0=0} \equiv 0$, we can write $\delta_u = -\rho_0\varepsilon_0$ with ρ_0 a smooth positive function. Thus, setting $\lambda'(\mu) := \lambda(\mu) + \min\big(\lambda(\mu), 1\big)$,

$$\alpha_1 := \begin{cases} -\frac{2(a+2)(b-1)}{(a+1)(b-2)} & \text{if } a < -1, \\ -1 & \text{if } a > -1, \end{cases} \qquad \alpha_2 := \begin{cases} -1 & \text{if } a < -1, \\ -\frac{a(b-1)}{2(a+1)b} & \text{if } a > -1, \end{cases} \text{ and } \rho_1 := \begin{cases} \kappa_{11} & \text{if } a > -1, \\ \kappa_{12} & \text{if } a < -1, \end{cases}$$

we can recap the whole statement of Proposition 3.2 as

$$-\mathscr{D}_u(s;\mu) = \varepsilon_0(\rho_0 + \star s^\lambda + \star s^{\lambda'}) + \varepsilon_+(\kappa_{01}s^\lambda + \star s^{\lambda'}) + (\alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 + \rho_2)\rho_1 s^{\lambda'} + \mathcal{F}_L^\infty(\mu_0), \tag{16}$$

where $\star$ are unspecified smooth functions on μ, $\rho_2 = \rho_2(a,b,\varepsilon_1,\varepsilon_2) = \mathrm{o}(\|(\varepsilon_1,\varepsilon_2)\|)$ and $L = \lambda'(\mu_0) + \delta'$ for some $\delta' > 0$ small enough. We remark that κ_{01}, κ_{11} and κ_{21} are smooth strictly positive functions given in Proposition 3.2. Thus ρ_2 is a smooth strictly positive function as well.

On the other hand, from (14) we get that $\alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 = \alpha_+\varepsilon_+ + \alpha_-\varepsilon_-$ with

$$\alpha_\pm := \frac{1}{2}\left(\alpha_2 \pm \frac{\alpha_1}{2}\sqrt{\frac{a(b-2)}{b(a+2)}}\right) = \begin{cases} \frac{1}{2}\left(-1 \mp \frac{(a+2)(b-1)}{(a+1)(b-2)}\sqrt{\frac{a(b-2)}{b(a+2)}}\right) & \text{if } a < -1, \\ \frac{1}{4}\left(-\frac{a(b-1)}{(a+1)b} \mp \sqrt{\frac{a(b-2)}{b(a+2)}}\right) & \text{if } a > -1. \end{cases} \tag{17}$$

Hence, since $\nu = \Phi(a,b,\varepsilon_0,\varepsilon_1,\varepsilon_2) := (\varepsilon_0,\varepsilon_+,\varepsilon_-,c_+,c_-)$ is a smooth change of coordinates in a neighbourhood U of μ_0 and $\big(\rho_2 \circ \Phi^{-1}\big)(\nu) = \underline{\rho}_2(\varepsilon_+,\varepsilon_-,c_+,c_-) = \mathrm{o}(\|(\varepsilon_+,\varepsilon_-)\|)$, the application of Lemma 3.4 yields

$$\big(\alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 + \rho_2(\mu)\big)\big|_{\mu=\Phi^{-1}(\nu)} = (\alpha_+ + \star)\varepsilon_+ + (\alpha_- + \varepsilon_-\eta_1)\varepsilon_- = \star\varepsilon_+ + (\alpha_- + \varepsilon_-\eta_1)\varepsilon_- \tag{18}$$

with $\eta_1 = \eta_1(\varepsilon_-, c_+, c_-)$. Here, and in what follows, for the sake of shortness, given a function $h = h(\mu)$ we denote $\underline{h} = \underline{h}(\nu) = h(\mu)|_{\mu=\Phi^{-1}(\nu)}$. Following this convention, from (16) and (18) we get

$$-\mathscr{D}_u(s;\mu)|_{\mu=\Phi^{-1}(\nu)} = \varepsilon_0(\underline{\rho}_0 + \star s^{\underline{\lambda}} + \star s^{\underline{\lambda}'}) + \varepsilon_+(\underline{\kappa}_{01} s^{\underline{\lambda}} + \star s^{\underline{\lambda}'}) + \varepsilon_-(\alpha_- + \varepsilon_-\eta_1)\underline{\rho}_1 s^{\underline{\lambda}'} + r(s;\nu),$$

where, setting $\nu_0 := \Phi(\mu_0)$ and applying assertion (h) in Lemma A.7, $r \in \mathcal{F}_L^\infty(\nu_0)$. Note that, by Lemma 3.3, if $\mu \in Z_0 = \{\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 0\}$ then $\mathscr{D}_u(s;\mu) \equiv 0$. Thus, since $\Phi(Z_0) = \{\varepsilon_0 = \varepsilon_+ = \varepsilon_- = 0\}$, we get that $r(s;\nu)|_{\varepsilon_0=\varepsilon_+=\varepsilon_-=0} \equiv 0$. By applying Lemma A.10 this implies that the remainder can be written as $r = \varepsilon_0 r_0 + \varepsilon_+ r_1 + \varepsilon_- r_2$ with $r_i \in \mathcal{F}_L^\infty(\nu_0)$. Consequently

$$-\mathscr{D}_u(s;\Phi^{-1}(\nu)) = \varepsilon_0\big(\underline{\rho}_0 + \star s^{\underline{\lambda}} + \star s^{\underline{\lambda}'} + r_0(s;\nu)\big) + \varepsilon_+\big(\underline{\kappa}_{01} s^{\underline{\lambda}} + \star s^{\underline{\lambda}'} + r_1(s;\nu)\big) + \varepsilon_-\big((\alpha_- + \varepsilon_-\eta_1)\underline{\rho}_1 s^{\underline{\lambda}'} + r_2(s;\nu)\big).$$

Furthermore, by Lemma 3.3 again, if $\mu \in Z_1 = \{a + b = \varepsilon_0 = 2\varepsilon_1 + \varepsilon_2 = 0\}$ then $\mathscr{D}_u(s;\mu) \equiv 0$. Thus, since one can easily check that $\Phi(Z_1) = \{\varepsilon_0 = \varepsilon_+ = c_- = 0\}$, we can assert that

$$(\alpha_- + \varepsilon_-\eta_1)\underline{\rho}_1 s^{\underline{\lambda}'} + r_2(s;\nu)\Big|_{\varepsilon_0=\varepsilon_+=c_-=0} \equiv 0.$$

Since $\rho_1(\mu_0) > 0$ and one can verify using (17) that $\alpha_- = c_-\eta_2$ with $\eta_2(\nu_0) > 0$, the above identity implies $\eta_1(\varepsilon_-, c_+, c_-)|_{c_-=0} \equiv 0$ and $r_2(s;\nu)|_{\varepsilon_0=\varepsilon_+=c_-=0} \equiv 0$. Accordingly $\eta_1(\varepsilon_-, c_+, c_-) = c_-\eta_3(\varepsilon_-, c_+, c_-)$ and, by Lemma A.10 once again, $r_2 = \varepsilon_0 r_3 + \varepsilon_+ r_4 + c_- r_5$ with $r_i \in \mathcal{F}_L^\infty(\nu_0)$. Consequently

$$-\mathscr{D}_u(s;\Phi^{-1}(\nu)) = \varepsilon_0\big(\underline{\rho}_0 + \star s^{\underline{\lambda}} + \star s^{\underline{\lambda}'} + \bar{r}_0(s;\nu)\big) + \varepsilon_+\big(\underline{\kappa}_{01} s^{\underline{\lambda}} + \star s^{\underline{\lambda}'} + \bar{r}_1(s;\nu)\big) + c_-\varepsilon_-\big(\eta_4 s^{\underline{\lambda}'} + r_5(s;\nu)\big).$$

where the new remainders $\bar{r}_0 = r_0 + r_3$ and $\bar{r}_1 = r_1 + r_4$ also belong to $\mathcal{F}_L(\nu_0)$ and $\eta_4 := (\eta_2 + \varepsilon_-\eta_3)\underline{\rho}_1$ satisfies $\eta_4(\nu_0) = (\eta_2\underline{\rho}_1)(\nu_0) > 0$. By applying Lemma A.7 we can take $\delta > 0$ small enough in order that the functions $s^{\underline{\lambda}}$, $s^{\underline{\lambda}'}$, $s^{\underline{\lambda}'-\underline{\lambda}}$, $s^{-\underline{\lambda}}\bar{r}_1$ and $s^{-\underline{\lambda}'} r_5$ belong to $\mathcal{F}_\delta(\nu_0)$. In doing so we obtain

$$-\mathscr{D}_u(s;\Phi^{-1}(\nu)) = \varepsilon_0\big(\underline{\rho}_0 + \mathcal{F}_\delta(\nu_0)\big) + \varepsilon_+ s^{\underline{\lambda}}\big(\underline{\kappa}_{01} + \mathcal{F}_\delta(\nu_0)\big) + c_-\varepsilon_- s^{\underline{\lambda}'}\big(\eta_4 + \mathcal{F}_\delta(\nu_0)\big).$$

Since $\nu = (\varepsilon_0, \varepsilon_+, \varepsilon_-, c_+, c_-)$, from this expression we obtain (15) by renaming the unit functions. This completes the proof. ■

Proof of Theorem B. We prove first the assertion with regard to the hemicycle Γ_u. By Lemma 3.1 it suffices to consider the quadratic 5-parameter perturbation given in (13). We set $\mu_0 = (a_0, b_0, 0, 0, 0)$ and note that the limit cycles of X_μ that are close to Γ_u in Hausdorff sense are in one to one correspondence with the isolated positive zeroes of

$$\mathscr{D}_u(s;\mu) = D_+^u(s;\mu) - D_-^u(s;\mu),$$

see Figure 7. That being said, by applying Theorem 3.5 we know that there exist a neighbourhood U of μ_0 and $\delta > 0$ small enough such that $\nu := \Phi(\mu) = (\varepsilon_0, \varepsilon_+, \varepsilon_-, c_+, c_-)$ is a local change of coordinates in U and

$$\mathscr{D}_u(s;\mu)|_{\mu=\Phi^{-1}(\nu)} = -\nu_1\big(\kappa_1 + \mathcal{F}_\delta^\infty(\nu_0)\big) - \nu_2 s^{\underline{\lambda}(\nu)}\big(\kappa_2 + \mathcal{F}_\delta^\infty(\nu_0)\big) - \nu_3\nu_5 s^{\underline{\lambda}'(\nu)}\big(\kappa_3 + \mathcal{F}_\delta^\infty(\nu_0)\big), \tag{19}$$

where $\nu_0 = \Phi(\mu_0)$, $\kappa_i(\nu_0) > 0$ and $\underline{\lambda}' = \underline{\lambda} + \min(\underline{\lambda}, 1)$.

Recall on the other hand, see Lemma 3.3, that $\mathscr{D}_u(s;\mu) \equiv 0$ if, and only if, $\mu \in Z_0 \cup Z_1$ where

$$Z_0 = \{\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 0\} \text{ and } Z_1 = \{a + b = \varepsilon_0 = 2\varepsilon_1 + \varepsilon_2 = 0\}.$$

One can check in this respect that $\Phi(Z_0 \cup Z_1) = \{\nu_1 = \nu_2 = \nu_3\nu_5 = 0\}$. Taking this into account, and the fact that $\Phi(\mu_0) = \nu_0$, we claim that there exist $s_0 > 0$ and an open ball $B_r(\nu_0)$ of radius $r > 0$

centered ν_0 such that (19) has at most two zeros on $(0,s_0)$, counted with multiplicities, for all ν inside $V:=B_r(\nu_0)\cap\{\nu_1^2+\nu_2^2+(\nu_3\nu_5)^2\neq 0\}$. This will imply, see Definition 2.3, that

$$\mathrm{Cycl}\big((\Gamma_u,X_{\mu_0}),X_\mu\big)\leqslant \mathscr{Z}_0\big(\mathscr{D}_u(\,\cdot\,;\mu),\mu_0\big)=\mathscr{Z}_0\big(\mathscr{D}_u(\,\cdot\,;\Phi^{-1}(\nu)),\nu_0\big)\leqslant 2.$$

In order to prove the claim we note first that, due to $\lim_{s\to 0}\big(\kappa_1(\nu)+\mathcal{F}_\delta^\infty(\nu_0)\big)=\kappa_1(\nu)\neq 0$ uniformly for $\nu\approx\nu_0$, we can take $r>0$ and $s_0>0$ small enough such that

$$\mathscr{R}_0(s;\nu):=-\frac{\mathscr{D}_u(s;\mu)|_{\mu=\Phi^{-1}(\nu)}}{\kappa_1+\mathcal{F}_\delta^\infty(\nu_0)}=\nu_1+\nu_2 s^{\underline{\lambda}(\nu)}\big(\kappa_4+\mathcal{F}_\delta^\infty(\nu_0)\big)+\nu_3\nu_5 s^{\underline{\lambda}'(\nu)}\big(\kappa_5+\mathcal{F}_\delta^\infty(\nu_0)\big)$$

is well defined for all $s\in(0,s_0)$ and $\nu\in B_r(\nu_0)$ and has exactly the same number of zeros, counted with multiplicities, as $\mathscr{D}_u(s;\Phi^{-1}(\nu))$. Accordingly $\mathscr{Z}_0\big(\mathscr{R}_0(\,\cdot\,;\nu),\nu_0\big)=\mathscr{Z}_0\big(\mathscr{D}_u(\,\cdot\,;\mu),\mu_0\big)$. We note that the second equality above follows from (19) by applying Lemma A.7 and that $\kappa_4:=\kappa_2/\kappa_1$ and $\kappa_5:\kappa_3/\kappa_1$ are strictly positive smooth functions. If $\nu\in V$ verifies $\nu_2=\nu_3\nu_5=0$ then $\nu_1\neq 0$ and, consequently, $\mathscr{R}_0(s;\nu)\neq 0$. This remark shows the validity of the claim for all $\nu\in V$ such that $\nu_2=\nu_3\nu_5=0$. To study the other cases we apply the so-called derivation-division algorithm. To this end we first observe that, by Lemma A.7 again,

$$\partial_s\mathscr{R}_0(s;\nu)=\nu_2 s^{\underline{\lambda}-1}\big(\underline{\lambda}\kappa_4+\mathcal{F}_\delta^\infty(\nu_0)\big)+\nu_3\nu_5 s^{\underline{\lambda}'-1}\big(\underline{\lambda}'\kappa_5+\mathcal{F}_\delta^\infty(\nu_0)\big)$$

and

$$\mathscr{R}_1(s;\nu):=\frac{\partial_s\mathscr{R}_0(s;\nu)}{s^{\underline{\lambda}-1}\big(\underline{\lambda}\kappa_4+\mathcal{F}_\delta^\infty(\nu_0)\big)}=\nu_2+\nu_3\nu_5 s^{\underline{\lambda}'-\underline{\lambda}}\big(\kappa_6+\mathcal{F}_\delta^\infty(\nu_0)\big),$$

where $\kappa_6(\nu_0)>0$. Note that $\lim_{s\to 0^+}\big(\underline{\lambda}\kappa_4(\nu)+\mathcal{F}_\delta^\infty(\nu_0)\big)=\underline{\lambda}\kappa_4(\nu)\neq 0$ uniformly for $\nu\approx\nu_0$. Therefore, by reducing $r>0$ and $s_0>0$ if necessary, $\mathscr{R}_1(s;\nu)$ is well defined for all $s\in(0,s_0)$ and $\nu\in B_r(\nu_0)$ and has exactly the same number of zeros, counted with multiplicities, as $\partial_s\mathscr{R}_0(s;\nu)$. If $\nu\in V$ verifies $\nu_3\nu_5=0$ then we can suppose that $\nu_2\neq 0$ (otherwise we end up in the previous case) and, consequently, $\mathscr{R}_1(s;\nu)\neq 0$. Hence, by applying Rolle's Theorem, the claim follows in this case. So far we have proved the validity of the claim for all $\nu\in V$ such that $\nu_3\nu_5=0$. To study the case $\nu_3\nu_5\neq 0$ we apply Lemma A.7 once again to obtain

$$\mathscr{R}_2(s;\nu):=\partial_s\mathscr{R}_1(s;\nu)=\nu_3\nu_5 s^{\underline{\lambda}'-\underline{\lambda}-1}\big(\kappa_7+\mathcal{F}_\delta^\infty(\nu_0)\big)$$

with $\kappa_5=(\underline{\lambda}'-\underline{\lambda})\kappa_6\neq 0$ for $\nu\approx\nu_0$. Exactly as before, by reducing $r>0$ and $s_0>0$ if necessary, we have that $\kappa_7(\nu)+\mathcal{F}_\delta^\infty(\nu_0)\neq 0$ for all $\nu\in B_r(\nu_0)$ and $s\in(0,s_0)$. Therefore $\mathscr{R}_2(s;\nu)\neq 0$ for all $\nu\in V$ with $\nu_3\nu_5\neq 0$ and $s\in(0,s_0)$ and the claim follows in this case by applying twice Rolle's Theorem. This exhausts all the possible cases for $\nu\in V$ and completes the proof of the claim. Accordingly $\mathrm{Cycl}\big((\Gamma_u,X_{\mu_0}),X_\mu\big)\leqslant 2$.

The fact that $\mathrm{Cycl}\big((\Gamma_u,X_{\mu_0}),X_\mu\big)\geqslant 2$ is also a consequence of (19). Indeed, by applying Proposition A.12 we can take a sequence $\lim_{n\to\infty}\hat{\nu}_n=\nu_0$ with $\hat{\nu}_n\in\Phi(U)\cap\{\nu_1=\nu_2=0\text{ and }\nu_3\nu_5\neq 0\}$ such that, setting $\hat{\mu}_n:=\Phi^{-1}(\hat{\nu}_n)$, we have $\mathrm{Cycl}\big((\Gamma_u,X_{\hat{\mu}_n}),X_\mu\big)\geqslant 2$ for all $n\in\mathbb{N}$. Since $\lim_{n\to\infty}\hat{\mu}_n=\mu_0$ this clearly implies that $\mathrm{Cycl}\big((\Gamma_u,X_{\mu_0}),X_\mu\big)\geqslant 2$, as desired.

So far we have proved that the cyclicity of Γ_u when we perturb (7) inside the whole family of quadratic differential systems is exactly 2. In order to show this for Γ_ℓ we use an orbital symmetry that preserves the two-parameter family (7) and interchanges Γ_ℓ with Γ_u. More concretely, we take $\phi(x,y)=(\eta x,-\eta^2 y)$ with $\eta:=\sqrt{\frac{b_0}{2-b_0}}$. Then one can verify that the coordinate change $(\bar{x},\bar{y})=\phi(x,y)$, together with the time reparametrization $\bar{t}=\eta^{-1}t$, induce the parameter change $(\bar{a}_0,\bar{b}_0)=(a_0,2-b_0)$ in the family (7). Due to $\phi(\Gamma_\ell)=\Gamma_u$, the result follows because we have already proved its validity for Γ_u. This completes the proof of the result. ∎

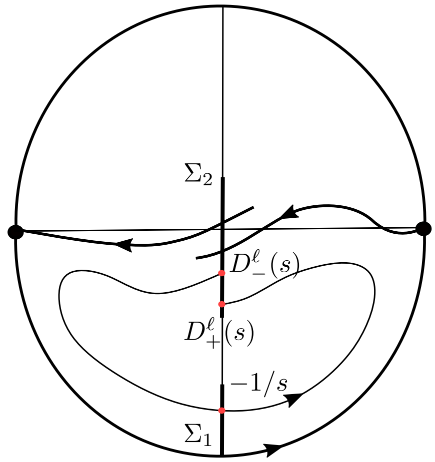

Figure 8: Dulac maps $D^\ell_\pm$ to define $\mathscr{D}_\ell(s;\mu) = D^\ell_+(s;\mu) - D^\ell_-(s;\mu)$. The points in red are $(0, D^\ell_\pm(s))$ and $(0,-1/s)$.

4 Proof of Theorem C

Lemma 4.1. *For each $b \in (0,2)$, define the linear map $\phi(x,y) = (\eta_b x, -\eta_b^2 y)$ with $\eta_b := \sqrt{\frac{b}{2-b}}$ and consider the vector field X_μ in (13). Then $\phi_\star(X_\mu) = \eta_b^{-1} X_{\sigma(\mu)}$ with $\sigma(a,b,\varepsilon_0,\varepsilon_1,\varepsilon_2) = (a, 2-b, -\eta_b^3\varepsilon_0, \eta_b\varepsilon_1, -\varepsilon_2/\eta_b)$.*

Proof. This follows by an easy computation and it is left to the reader. ■

The previous result will enable us to study the limit cycles bifurcating from Γ_ℓ by taking advantage of Theorem 3.5, which is addressed to the ones bifurcating from Γ_u. To this end we take two transverse sections on $x=0$, Σ_1 and Σ_2, parametrized by $s \mapsto (0,-1/s)$ with $s \in (0,\delta)$ and $s \mapsto (0,s)$ with $s \in (-\delta,\delta)$, respectively. Then, see Figure 8, we consider the Dulac map $D^\ell_+(\,\cdot\,;\mu)$ for X_μ from Σ_1 to Σ_2 and the Dulac map $D^\ell_-(\,\cdot\,;\mu)$ for $-X_\mu$ from Σ_1 to Σ_2 and define

$$\mathscr{D}_\ell(s;\mu) := D^\ell_+(s;\mu) - D^\ell_-(s;\mu).$$

We remark that, according to the parametrization of Σ_1, the function $\mathscr{D}_\ell(s;\mu)$ is defined for positive s. Taking these definitions into account we now prove the following result. With regard to its statement we stress that the change of parameters $\nu = \Phi(\mu)$ is the same as the one given in Theorem 3.5, cf. (14).

Corollary 4.2. *Given any $\mu_0 = (a_0, b_0, 0, 0, 0)$ with $a_0 \in (-2,0)\setminus\{-1\}$ and $b_0 \in (0,2)$, there exist a neighbourhood U of μ_0 in $\mathbb{R}^5$ and $\delta > 0$ such $\nu = \Phi(\mu) := (\varepsilon_0, \varepsilon_+, \varepsilon_-, c_+, c_-)$ is a local change of coordinates in U and we can write*

$$\mathscr{D}_u(s;\mu)|_{\mu=\Phi^{-1}(\nu)} = -\nu_1\hat{g}_1(s;\nu) - \nu_3\hat{g}_2(s;\nu) - \nu_2\nu_4\hat{g}_3(s;\nu),$$

where, setting $\nu_0 = \Phi(\mu_0) = (0,0,0,\nu_4^0,\nu_5^0)$,

(a) $\hat{g}_1(s;\nu) = \hat{\kappa}_1(\nu) + \mathcal{F}^\infty_\delta(\nu_0)$,

(b) $\hat{g}_2(s;\nu) = s^{\underline{\lambda}(\nu)}\big(\hat{\kappa}_2(\nu) + \mathcal{F}^\infty_\delta(\nu_0)\big)$ *where* $\underline{\lambda}(\nu)|_{\nu=\Phi(\mu)} = -\frac{a+2}{a}$, *and*

(c) $\hat{g}_3(s;\nu) = s^{\underline{\lambda}'(\nu)}\big(\hat{\kappa}_3(\nu) + \mathcal{F}^\infty_\delta(\nu_0)\big)$ *where* $\underline{\lambda}'(\nu) = \underline{\lambda}(\nu) + \min\big(\underline{\lambda}(\nu), 1\big)$.

Moreover $\hat{\kappa}_1$, $\hat{\kappa}_2$ and $\hat{\kappa}_3$ are smooth strictly positive functions on $\Phi(U)$.

Proof. By applying Lemma 4.1 (and following the notation given in its statement) one can easily show that $D_\pm^\ell(s;\mu) = -\eta_b^{-2} D_\pm^u\big(\eta_b^{-2}s;\sigma(\mu)\big)$. Thus $\mathscr{D}_\ell(s;\mu) = -\eta_b^{-2}\mathscr{D}_u\big(\eta_b^{-2}s;\sigma(\mu)\big)$ and, consequently,

$$\begin{aligned}\mathscr{D}_\ell(s;\mu)|_{\mu=\Phi^{-1}(\nu)} &= -\,\eta_b^{-2}\mathscr{D}_u\big(\eta_b^{-2}s;\sigma(\mu)\big)\big|_{\mu=\Phi^{-1}(\nu)}\\ &= -\hat{\eta}_\nu^{-2}\mathscr{D}_u\big(\hat{\eta}_\nu^{-2}s;\sigma(\Phi^{-1}(\nu))\big)\\ &= -\hat{\eta}_\nu^{-2}\mathscr{D}_u\big(\hat{\eta}_\nu^{-2}s;\mu\big)|_{\mu=\Phi^{-1}(\hat{\sigma}(\nu))}\end{aligned}$$

where in the second equality we set $\hat{\eta}_\nu := \eta_b|_{\mu=\Phi^{-1}(\nu)} = \sqrt{\frac{2+\nu_4-\nu_5}{2-\nu_4+\nu_5}}$ and in the third one $\hat{\sigma} := \Phi\circ\sigma\circ\Phi^{-1}$. Some computations show that

$$\hat{\sigma}(\nu) = (-\hat{\eta}_\nu^3\nu_1, -\nu_3/\hat{\eta}_\nu, -\nu_2/\hat{\eta}_\nu, \nu_5, \nu_4).$$

Therefore, from the equality (15) in Theorem 3.5, we obtain that

$$\begin{aligned}\mathscr{D}_\ell(s;\mu)|_{\mu=\Phi^{-1}(\nu)} &= -\hat{\eta}_\nu^{-2}\mathscr{D}_u\big(\hat{\eta}_\nu^{-2}s;\mu\big)|_{\mu=\Phi^{-1}(\hat{\sigma}(\nu))}\\ &= -\hat{\eta}_\nu^{-2}\Big(\hat{\eta}_\nu^3\nu_1 g_1\big(\hat{\eta}_\nu^{-2}s;\hat{\sigma}(\nu)\big) + \nu_3/\hat{\eta}_\nu g_2\big(\hat{\eta}_\nu^{-2}s;\hat{\sigma}(\nu)\big) + \nu_2\nu_4/\hat{\eta}_\nu g_3\big(\hat{\eta}_\nu^{-2}s;\hat{\sigma}(\nu)\big)\Big),\end{aligned}$$

and so the result follows setting

$$\hat{g}_1(s;\nu) := \hat{\eta}_\nu g_1\big(\hat{\eta}_\nu^{-2}s;\hat{\sigma}(\nu)\big),\ \hat{g}_2(s;\nu) := \hat{\eta}_\nu^{-3} g_2\big(\hat{\eta}_\nu^{-2}s;\hat{\sigma}(\nu)\big) \text{ and } \hat{g}_3(s;\nu) := \hat{\eta}_\nu^{-3} g_3\big(\hat{\eta}_\nu^{-2}s;\hat{\sigma}(\nu)\big),$$

which satisfy conditions (a), (b) and (c) in the statement due to $\underline{\lambda}\circ\hat{\sigma} = \underline{\lambda}$, $\hat{\eta}_\nu > 0$ and assertion (h) of Lemma A.7. This concludes the proof of the result. ∎

Proof of Theorem C. By applying Lemma 3.1 it suffices to consider the quadratic 5-perturbation $\{X_\mu\}$ given in (13). To begin with let us take $\mu_0 = (a_0, b_0, 0, 0, 0)$ with $a_0 \neq -1$ and note that then by Theorem 3.5 and Corollary 4.2, respectively, we obtain that

$$\begin{aligned}\mathscr{R}_u(s;\nu) := -\frac{\mathscr{D}_u(s;\mu)|_{\mu=\Phi^{-1}(\nu)}}{g_1(s;\nu)} &= \nu_1 + \nu_2\frac{g_2(s;\nu)}{g_1(s;\nu)} + \nu_3\nu_5\frac{g_3(s;\nu)}{g_1(s;\nu)}\\ &= \nu_1 + \nu_2 h_2(s;\nu) + \nu_3\nu_5 h_3(s;\nu)\end{aligned}\tag{20}$$

and

$$\begin{aligned}\mathscr{R}_\ell(s;\nu) := -\frac{\mathscr{D}_\ell(s;\mu)|_{\mu=\Phi^{-1}(\nu)}}{\hat{g}_1(s;\nu)} &= \nu_1 + \nu_3\frac{\hat{g}_2(s;\nu)}{\hat{g}_1(s;\nu)} + \nu_2\nu_4\frac{\hat{g}_3(s;\nu)}{\hat{g}_1(s;\nu)}\\ &= \nu_1 + \nu_3\hat{h}_2(s;\nu) + \nu_2\nu_4\hat{h}_3(s;\nu),\end{aligned}\tag{21}$$

where by applying Lemma A.7 we have that

$$h_2 := g_2/g_1 = s^{\underline{\lambda}(\nu)}\big(\kappa_4(\nu) + \mathcal{F}_\delta^\infty(\nu_0)\big) \text{ and } h_3 := g_3/g_1 = s^{\underline{\lambda}'(\nu)}\big(\kappa_5(\nu) + \mathcal{F}_\delta^\infty(\nu_0)\big)\tag{22}$$

with $\kappa_i(\nu_0) > 0$ and

$$\hat{h}_2 := \hat{g}_2/\hat{g}_1 = s^{\underline{\lambda}(\nu)}\big(\hat{\kappa}_4(\nu) + \mathcal{F}_\delta^\infty(\nu_0)\big) \text{ and } \hat{h}_3 := \hat{g}_3/\hat{g}_1 = s^{\underline{\lambda}'(\nu)}\big(\hat{\kappa}_5(\nu) + \mathcal{F}_\delta^\infty(\nu_0)\big)\tag{23}$$

with $\hat{\kappa}_i(\nu_0) > 0$. Note that the limit cycles of X_μ that are close to Γ_u (respectively, Γ_ℓ) in Hausdorff sense are in one to one correspondence with the isolated positive zeroes of $\mathscr{D}_u(\,\cdot\,;\mu)$ (respectively, $\mathscr{D}_\ell(\,\cdot\,;\mu)$). In turn, those zeroes are in one to one correspondence with the ones of $\mathscr{R}_u(\,\cdot\,;\nu)$ and $\mathscr{R}_\ell(\,\cdot\,;\nu)$, respectively, where $\nu=\Phi(\mu)$.

We claim first that $\mathrm{Cycl}\big((\{\Gamma_u,\Gamma_\ell\},X_{\mu_0}),X_\mu\big)\leqslant 3$. We prove it by contradiction. If the claim is false then, since we know by Theorem B that $\mathrm{Cycl}((\Gamma_u,X_{\mu_0}),X_\mu)=\mathrm{Cycl}((\Gamma_\ell,X_{\mu_0}),X_\mu)=2$, by applying Rolle's Theorem there would exist three sequences $s_n\to 0^+$, $s_n'\to 0^+$ and $\nu_n\to\nu_0:=\Phi(\mu_0)$ such that $\partial_s\mathscr{R}_u(s_n;\nu_n)=\partial_s\mathscr{R}_\ell(s_n';\nu_n)=0$ for all n. On the other hand, by Lemma A.7 again, from (22) we get that

$$\lim_{s\to 0^+}\frac{\partial_s h_3(s;\nu)}{\partial_s h_2(s;\nu)}=0 \text{ uniformly on } \nu\approx\nu_0.$$

Then from (20) we obtain that $\partial_s\mathscr{R}_u(s_n;\nu_n)=-\nu_2\partial_s h_2(s_n;\nu)-\nu_3\nu_5\partial_s h_3(s_n;\nu)\big|_{\nu=\nu_n}=0$ for all n and, consequently,

$$\left.\frac{\nu_2}{\nu_3\nu_5}\right|_{\nu=\nu_n}=-\frac{\partial_s h_3(s_n;\nu_n)}{\partial_s h_2(s_n;\nu_n)}\to 0 \text{ as } n\to\infty.$$

Therefore $\lim_{n\to\infty}\frac{\nu_2}{\nu_3\nu_5}\big|_{\nu=\nu_n}=0$. Exactly the same way, but using (23) and that $\partial_s\mathscr{R}_\ell(s_n';\nu_n)=0$ for all n, we get that $\lim_{n\to\infty}\frac{\nu_3}{\nu_2\nu_4}\big|_{\nu=\nu_n}=0$. The combination of both limits implies that $\frac{1}{\nu_4\nu_5}\big|_{\nu=\nu_n}$ tends to 0 as $n\to\infty$, which is a contradiction because $\lim_{n\to\infty}\nu_n=\nu_0\in\mathbb{R}^5$. This proves the claim.

In order to proceed we take $\varepsilon>0$ and $s_0>0$ small enough such that the functions $h_i(s;\nu)$ and $\hat{h}_i(s;\nu)$ for $i=1,2$ are strictly positive for all $s\in(0,s_0)$ and $\nu\in B_\varepsilon(\nu_0)$.

We claim next that $\mathrm{Cycl}((\{\Gamma_u,\Gamma_\ell\},X_{\mu_0}),X_\mu)\geqslant 3$ for all $(a_0,b_0)\in(-2,0)\times(0,2)$ with $a_0\neq -1$ verifying that $a_0+b_0\leqslant 0$ or $a_0+2-b_0\leqslant 0$. Let us assume for instance that $a_0+b_0\leqslant 0$ (the other case follows verbatim). To this end recall, see (14), that $\nu_0=\Phi(\mu_0)=(0,0,0,a_0+2-b_0,a_0+b_0)$ and so the fifth component of ν_0 is not positive. That being said we take $\bar\nu\in B_\varepsilon(\nu_0)\cap\{\nu_1=\nu_2=0,\nu_3\neq 0,\nu_5<0\}$ and $s_1\in(0,s_0)$ in order that $\bar\nu_3\mathscr{R}_u(s_1;\bar\nu)>0$ and $\bar\nu_3\mathscr{R}_\ell(s_1;\bar\nu)<0$, see (20) and (21), respectively. Next, by continuity, we can take $\hat\nu\in B_\varepsilon(\nu_0)\cap\{\nu_1=0,\nu_2\nu_3>0\}$ close enough to $\bar\nu$ in order to have

$$\hat\nu_3\bar\nu_3>0,\quad \mathscr{R}_u(s_1;\hat\nu)\mathscr{R}_u(s_1;\bar\nu)>0 \text{ and } \mathscr{R}_\ell(s_1;\hat\nu)\mathscr{R}_\ell(s_1;\bar\nu)>0.$$

We take then $s_2\in(0,s_1)$ small enough such that, on account of (20) and (22), $\hat\nu_2\mathscr{R}_u(s_2;\hat\nu)<0$. Finally, by continuity again, we choose $\nu^\star\in B_\varepsilon(\nu_0)\cap\{\nu_1\nu_2<0\}$ close enough to $\hat\nu$ such that

$$\begin{aligned}
&\mathscr{R}_u(s_1;\nu^\star)\mathscr{R}_u(s_1;\hat\nu)>0 &&\nu_2^\star\hat\nu_2>0\\
&\mathscr{R}_\ell(s_1;\nu^\star)\mathscr{R}_\ell(s_1;\hat\nu)>0 &&\nu_3^\star\hat\nu_3>0\\
&\mathscr{R}_u(s_2;\nu^\star)\mathscr{R}_u(s_2;\hat\nu)>0
\end{aligned}$$

Observe that we can also take $s_3\in(0,s_2)$ small enough such that, thanks to (20) and (21),

$$\nu_1^\star\mathscr{R}_u(s_3;\nu^\star)<0 \text{ and } \nu_1^\star\mathscr{R}_\ell(s_3;\nu^\star)<0.$$

Then $\mathscr{R}_\ell(s_1;\nu^\star)\mathscr{R}_\ell(s_3;\nu^\star)<0$ due to $\nu_2^\star\nu_3^\star>0$ and $\nu_1^\star\nu_2^\star<0$. Therefore, by Bolzano's Theorem, there exists $s_\ell\in(s_3,s_1)$ such that $\mathscr{R}_\ell(s_\ell;\nu^\star)=0$. On the other hand,

$$\mathscr{R}_u(s_3;\nu^\star)\mathscr{R}_u(s_2;\nu^\star)<0 \text{ and } \mathscr{R}_u(s_2;\nu^\star)\mathscr{R}_u(s_1;\nu^\star)<0$$

due to $\nu_1^\star\nu_2^\star<0$ and $\nu_2^\star\nu_3^\star>0$, respectively. Consequently, by applying Bolzano's Theorem again, there exist $s_u^1\in(s_3,s_2)$ and $s_u^2\in(s_2,s_1)$ such that $\mathscr{R}_u(s_u^1;\nu^\star)=\mathscr{R}_u(s_u^2;\nu^\star)=0$. Summing-up, we have proved that there exist $\nu^\star\in B_\varepsilon(\nu_0)$ and $s_\ell,s_u^1,s_u^2\in(0,s_0)$ with $s_u^1\neq s_u^2$ such that

$$\mathscr{R}_\ell(s_\ell;\nu^\star)=\mathscr{R}_u(s_u^1;\nu^\star)=\mathscr{R}_u(s_u^2;\nu^\star)=0.$$

Accordingly $\text{Cycl}((\{\Gamma_u,\Gamma_\ell\},X_{\mu_0}),X_\mu)\geqslant 3$ because $\nu_0=\Phi(\mu_0)$ and we can take $\varepsilon>0$ and $s_0>0$ arbitrarily small. This proves the claim. (For completeness let us note that the case $a_0+2-b_0\leqslant 0$ leads to the simultaneous bifurcation of one limit cycle from Γ_u and two from Γ_ℓ.)

Thanks to the claim we also have that $\text{Cycl}((\{\Gamma_u,\Gamma_\ell\},X_{\mu_0}),X_\mu)\geqslant 3$ for each $\mu_0=(a_0,b_0,0,0,0)$ with $(a_0,b_0)\in\{-1\}\times(0,2)$ because in any neighbourhood of such μ_0 there exist a parameter $\mu_\star$, not in $\{a=-1\}$, verifying that $\text{Cycl}((\{\Gamma_u,\Gamma_\ell\},X_{\mu_\star}),X_\mu)\geqslant 3$.

Our last task is to show that if $(a_0,b_0)\in\mathcal{K}_2$, i.e., $a_0+b_0>0$ and $a_0+2-b_0>0$ is verified, then $\text{Cycl}((\{\Gamma_u,\Gamma_\ell\},X_{\mu_0}),X_\mu)=2$. To this end, on account of

$$\text{Cycl}\big((\{\Gamma_u,\Gamma_\ell\},X_{\mu_0}),X_\mu\big)\geqslant\max\big\{\text{Cycl}\big((\Gamma_u,X_{\mu_0}),X_\mu\big),\text{Cycl}\big((\Gamma_\ell,X_{\mu_0}),X_\mu\big)\big\}=2,$$

it is clear that it suffices to prove that $\text{Cycl}\big((\{\Gamma_u,\Gamma_\ell\},X_{\mu_0}),X_\mu\big)\leqslant 2$. We shall bound this number by studying the positive zeros of $\mathscr{R}_u(s;\nu)$ and $\mathscr{R}_\ell(s;\nu)$, see (20) and (21), bifurcating from $s=0$ when ν tends to $\nu_0\in\{\nu_1=\nu_2=\nu_3=0,\nu_4>0\text{ and }\nu_5>0\}$. Recall here that $\nu_0=\Phi(\mu_0)=(0,0,0,a_0+2-b_0,a_0+b_0)$. On account of (22) and (23), respectively, the application of Lemma A.7 yields

$$h_2(s;\nu)=\kappa_4(\nu)\left(s\left(1+\mathcal{F}_\delta^\infty(\nu_0)\right)\right)^{\underline{\lambda}(\nu)}\text{ and }\hat{h}_2(s;\nu)=\hat{\kappa}_4(\nu)\left(s\left(1+\mathcal{F}_\delta^\infty(\nu_0)\right)\right)^{\underline{\lambda}(\nu)}.$$

Accordingly, by applying twice Lemma A.9 we deduce that

$$(t,\nu)=\Psi(s,\nu):=\big(h_2(s;\nu),\nu\big)\text{ and }(t,\nu)=\hat{\Psi}\big(s;\nu):=\big(\hat{h}_2(s;\nu),\nu\big)$$

are well defined changes of variables satisfying

$$\Psi^{-1}(t,\nu)=\big(\sigma((t/\kappa_4(\nu))^{1/\underline{\lambda}(\nu)};\nu),\nu\big)\text{ and }\hat{\Psi}^{-1}(t,\nu)=\big(\hat{\sigma}((t/\hat{\kappa}_4(\nu))^{1/\underline{\lambda}(\nu)};\nu),\nu\big),$$

where $\sigma(u;\nu):=u(1+\mathcal{F}_\delta^\infty(\nu_0))$ and $\hat{\sigma}(u;\nu):=u(1+\mathcal{F}_\delta^\infty(\nu_0))$. Our aim is to apply these changes of variables in (20) and (21), respectively. To this end note that, by Lemma A.7 once again, from (22) and (23) we get

$$\big(h_3\circ\Psi^{-1}\big)(t;\nu)=t^{\vartheta(\nu)}(\kappa(\nu)+f(t;\nu))\text{ and }\big(\hat{h}_3\circ\hat{\Psi}^{-1}\big)(t;\nu)=t^{\vartheta(\nu)}(\hat{\kappa}(\nu)+\hat{f}(t;\nu))$$

with $\vartheta(\nu):=\underline{\lambda}'(\nu)/\underline{\lambda}(\nu)=1+\min(1,1/\underline{\lambda}(\nu))>1$, κ and $\hat{\kappa}$ smooth positive functions, and $f,\hat{f}\in\mathcal{F}_{\delta_1}^\infty(\nu_0)$ for some $\delta_1>0$ small enough. Accordingly, from (20) and (21),

$$\bar{\mathscr{R}}_u(t;\nu):=(\mathscr{R}_u\circ\Psi^{-1})(t,\nu)=\nu_1+\nu_2 t+\nu_3\nu_5 t^{\vartheta(\nu)}(\kappa(\nu)+f(t;\nu))$$

and

$$\bar{\mathscr{R}}_\ell(t;\nu):=(\mathscr{R}_\ell\circ\hat{\Psi}^{-1})(t,\nu)=\nu_1+\nu_3 t+\nu_2\nu_4 t^{\vartheta(\nu)}(\hat{\kappa}(\nu)+\hat{f}(t;\nu)).$$

We are now in position to prove that $\text{Cycl}((\{\Gamma_u,\Gamma_\ell\},X_{\mu_0}),X_\mu)\leqslant 2$. By contradiction, if this number is greater than 2 then (by exchanging the subindices u and ℓ if necessary) for all $\varepsilon>0$ there would exist

$$(t_1,t_2,t_3,\nu)\in W_\varepsilon:=(0,\varepsilon)^3\times B_\varepsilon(\nu_0)\setminus\big\{t_1=t_2\text{ or }\nu_1=\nu_2=\nu_3=0\big\}$$

verifying that

$$\bar{\mathscr{R}}_u(t_1;\nu)=\bar{\mathscr{R}}_u(t_2;\nu)=\bar{\mathscr{R}}_\ell(t_3;\nu)=0.$$

These three equalities can be written as

$$\begin{pmatrix}1 & a & \nu_5 t_1^{\vartheta(\nu)}(\kappa(\nu)+f(t_1;\nu))\\ 1 & b & \nu_5 t_2^{\vartheta(\nu)}(\kappa(\nu)+f(t_2;\nu))\\ 1 & \nu_4 t_3^{\vartheta(\nu)}(\hat{\kappa}(\nu)+\hat{f}(t_3;\nu)) & c\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

A necessary condition for this to hold is that the determinant

$$D(t_1,t_2,t_3;\nu):=\begin{vmatrix} 1 & a & \nu_5 t_1^{\vartheta(\nu)}(\kappa(\nu)+f(t_1;\nu)) \\ 1 & b & \nu_5 t_2^{\vartheta(\nu)}(\kappa(\nu)+f(t_2;\nu)) \\ 1 & \nu_4 t_3^{\vartheta(\nu)}(\hat{\kappa}(\nu)+\hat{f}(t_3;\nu)) & c \end{vmatrix}$$

is equal to zero because $(t_1,t_2,t_3,\nu)\in W_\varepsilon$. An easy computation shows that we can write

$$\frac{D(t_1,t_2,t_3;\nu)}{t_2-t_1}=t_3\left(1-\nu_5 c^{\vartheta(\nu)-1}\left(\hat{\kappa}(\nu)+\hat{f}(t_3;\nu)\right)A_0(t_1,t_2;\nu)\right)+\nu_5 t_1 t_2 A_1(t_1,t_2;\nu), \tag{24}$$

where, for $i=0,1$,

$$\begin{aligned} A_i(t_1,t_2;\nu) &:= \frac{t_2^{\vartheta(\nu)-i}(\kappa(\nu)+f(t_2;\nu))-t_1^{\vartheta(\nu)-i}(\kappa(\nu)+f(t_1;\nu))}{t_2-t_1} \\ &= \frac{t_2^{\vartheta(\nu)-i}-t_1^{\vartheta(\nu)-i}}{t_2-t_1}\left(\kappa(\nu)+\frac{f_i\big(t_2^{\vartheta(\nu)-i};\nu\big)-f_i\big(t_1^{\vartheta(\nu)-i};\nu\big)}{t_2^{\vartheta(\nu)-i}-t_1^{\vartheta(\nu)-i}}\right), \end{aligned}$$

with $f_i(r;\nu):=rf\left(r^{\frac{1}{\vartheta(\nu)-i}};\nu\right)\in\mathcal{F}^\infty_{1+\delta_2}(\nu_0)$ for some $\delta_2>0$ small enough. By applying (twice) the Mean Value Theorem there exist $\alpha_i>0$ between t_1 and t_2, together with $\beta_i>0$ between $t_1^{\vartheta(\nu)-i}$ and $t_2^{\vartheta(\nu)-i}$, (depending both on t_1, t_2 and ν) such that

$$A_i(t_1,t_2;\nu)=\big(\vartheta(\nu)-i\big)\alpha_i^{\vartheta(\nu)-i-1}\big(\kappa(\nu)+\partial_r f_i(\beta_i;\nu)\big)\text{ for each } i=0,1.$$

On account of $\vartheta(\nu_0)>1$ and $\partial_r f_i\in\mathcal{F}^\infty_{\delta_2}(\nu_0)$ with $\delta_2>0$, we can assert that $A_0(t_1,t_2;\nu)$ tends to zero as $(t_1,t_2,\nu)\to(0^+,0^+,\nu_0)$ and that $A_1(t_1,t_2;\nu)>0$ on W_ϵ for $\epsilon>0$ small enough. Since $\nu_5>0$, from (24) we conclude that $(t_2-t_1)D(t_1,t_2,t_3;\nu)>0$ for all $(t_1,t_2,t_3,\nu)\in W_\epsilon$ with $\epsilon>0$ small enough. This contradicts $D(t_1,t_2,t_3;\nu)=0$ and so $\mathrm{Cycl}\big((\{\Gamma_u,\Gamma_\ell\},X_{\mu_0}),X_\mu\big)\leqslant 2$. This concludes the proof of the result. ∎

5 Proof of Theorem D

In this section we shall demonstrate Theorem D. However, prior to that, we shall give two general results regarding the notion of boundary cyclicity from inside, see Definition 1.7. Thus, otherwise explicitly stated, we consider a germ $\{X_\mu\}_{\mu\approx\mu_0}$ of an arbitrary analytic family of vector fields on $\mathbb{S}^2$. Given $U\subseteq\mathbb{S}^2$, we denote by $\mathcal{C}(U)$ the set of compact subsets $K\subset U$ and, as usual, $N_\varepsilon(U)$ stands for the open ε-neighbourhood of U. We also denote the set of limit periodic sets of the germ $\{X_\mu\}_{\mu\approx\mu_0}$ by $\mathcal{L}$, so that $\mathcal{L}\subset\mathcal{C}(\mathbb{S}^2)$, see Definition 1.2.

Lemma 5.1. *If U is an open subset of $\mathbb{S}^2$ then $\underline{\mathrm{Cycl}}_G^U\big((\partial U,X_{\mu_0}),X_\mu\big)\leqslant\mathrm{Cycl}_G\big((\partial U,X_{\mu_0}),X_\mu\big)$.*

Proof. Fix a natural number $c\leqslant\underline{\mathrm{Cycl}}_G^U\big((\partial U,X_{\mu_0}),X_\mu\big)$. For any $\rho>0$ the set $K_\rho:=\overline{U}\setminus N_\rho(\partial U)\in\mathcal{C}(U)$ verifies $U\setminus K_\rho\subset N_\rho(\partial U)$ and, on the other hand (see Definition 1.7), $\mathrm{Cycl}_G\big((U\setminus K_\rho,X_{\mu_0}),X_\mu\big)\geqslant c$. This means, recall Definition 1.6, that there exists $L_\rho\in\mathcal{C}(U\setminus K_\rho)$ for which $\mathrm{Cycl}_G\big((L_\rho,X_{\mu_0}),X_\mu\big)\geqslant c$, i.e., for all $\varepsilon,\delta>0$ there exists $\mu\in B_\delta(\mu_0)$ such that X_μ has at least c limit cycles inside $N_\varepsilon(L_\rho)\subset N_{\varepsilon+\rho}(\partial U)$. According to Definition 1.6 again, we conclude that $\mathrm{Cycl}_G\big((\partial U,X_{\mu_0}),X_\mu\big)\geqslant c$. If $\underline{\mathrm{Cycl}}_G^U\big((\partial U,X_{\mu_0}),X_\mu\big)$ is finite we can take $c=\underline{\mathrm{Cycl}}_G^U\big((\partial U,X_{\mu_0}),X_\mu\big)$ to obtain that $\mathrm{Cycl}_G\big((\partial U,X_{\mu_0}),X_\mu\big)\geqslant\underline{\mathrm{Cycl}}_G^U\big((\partial U,X_{\mu_0}),X_\mu\big)$, otherwise we easily deduce $\underline{\mathrm{Cycl}}_G^U\big((\partial U,X_{\mu_0}),X_\mu\big)=\infty=\mathrm{Cycl}_G\big((\partial U,X_{\mu_0}),X_\mu\big)$. ∎

Lemma 5.2. *If $K\in\mathcal{C}(\mathbb{S}^2)$ then $\mathrm{Cycl}_G\big((K,X_{\mu_0}),X_\mu\big)=\mathrm{Cycl}\big((\mathcal{L}(K),X_{\mu_0}),X_\mu\big)$, where $\mathcal{L}(K)=\mathcal{L}\cap\mathcal{C}(K)$.*

Proof. By Remark 1.1, the set $\{\gamma$ limit cycle of X_μ contained in $N_\varepsilon(K)\}$ contains

$$\{\gamma \text{ limit cycle of } X_\mu \text{ with } d_H(\gamma,\Gamma)<\varepsilon \text{ for some } \Gamma\in\mathcal{L}(K)\}.$$

Accordingly, on account of Definitions 1.4 and 1.6, it follows that

$$\mathrm{Cycl}_G\big((K,X_{\mu_0}),X_\mu\big)\geqslant \mathrm{Cycl}\big((\mathcal{L}(K),X_{\mu_0}),X_\mu\big).$$

Fix a natural number $c\leqslant \mathrm{Cycl}_G\big((K,X_{\mu_0}),X_\mu\big)$. Then, see Definition 1.6 again, for any $n\in\mathbb{N}$ there exists $\mu_n\in B_{1/n}(\mu_0)$ such that X_{μ_n} has at least c limit cycles $\gamma_n^1,\ldots,\gamma_n^c$ contained in $N_{1/n}(K)$. Since $\big(\mathcal{C}(\mathbb{S}^2),d_H\big)$ is compact (see Remark 1.1 again), by taking a subsequence we can assume that $\gamma_n^j\to\Gamma^j\in\mathcal{L}(K)$ as $n\to\infty$. Consequently, for each $\varepsilon,\delta>0$, there exists $n\in\mathbb{N}$ such that $\mu_n\in B_\delta(\mu_0)$ and $d_H(\gamma_n^j,\Gamma^j)<\varepsilon$. Therefore, see Definition 1.4, $\mathrm{Cycl}\big((\mathcal{L}(K),X_{\mu_0}),X_\mu\big)\geqslant c$. If $\mathrm{Cycl}_G\big((K,X_{\mu_0}),X_\mu\big)$ is finite then we can take $c=\mathrm{Cycl}_G\big((K,X_{\mu_0}),X_\mu\big)$ to conclude that $\mathrm{Cycl}\big((\mathcal{L}(K),X_{\mu_0}),X_\mu\big)\geqslant \mathrm{Cycl}_G\big((K,X_{\mu_0}),X_\mu\big)$, and the result follows. Otherwise one can easily show that

$$\mathrm{Cycl}_G\big((K,X_{\mu_0}),X_\mu\big)=\infty=\mathrm{Cycl}\big((\mathcal{L}(K),X_{\mu_0}),X_\mu\big),$$

and so the result follows as well. ∎

Proof of Theorem D. Let $\{X_\mu\}_{\mu\in\Lambda}$ be the whole quadratic family of vector fields and X_{μ_0} the vector field (7) with $(a_0,b_0)\in\{(-1,1),(-\frac{1}{2},\frac{1}{2}),(-\frac{1}{2},\frac{3}{2})\}$. Setting $U=\mathbb{R}^2\setminus\{y=0\}$, so that $\partial U=\Gamma_u\cup\Gamma_\ell$, we get

$$\underline{\mathrm{Cycl}}_G^U\big((\partial U,X_{\mu_0}),X_\mu\big)\overset{(1)}{\leqslant}\mathrm{Cycl}_G\big((U,X_{\mu_0}),X_\mu\big)\overset{(2)}{=}2<3\overset{(3)}{\leqslant}\mathrm{Cycl}\big((\{\Gamma_u,\Gamma_\ell\},X_{\mu_0}),X_\mu\big)$$
$$\overset{(4)}{=}\mathrm{Cycl}\big((\mathcal{L}(\partial U),X_{\mu_0}),X_\mu\big)\overset{(5)}{=}\mathrm{Cycl}_G\big((\partial U,X_{\mu_0}),X_\mu\big).$$

The inequality (1) follows from Definition 1.7 taking $K=\emptyset$. The equality (2) for $(a_0,b_0)=(-1,1)$ follows from [8, Theorem 11], for $(a_0,b_0)=(-\frac{1}{2},\frac{1}{2})$ follows from [21, Theorem 1.2] and for $(a_0,b_0)=(-\frac{1}{2},\frac{3}{2})$ is a consequence of the latter by applying Lemmas 3.1 and 4.1. The inequality (3) follows from Theorem C. The equality (4) is due to the fact that the only limit periodic sets inside $\partial U=\Gamma_u\cup\Gamma_\ell$ are Γ_u and Γ_ℓ. Finally, the equality (5) follows from Lemma 5.2. This proves the result. ∎

Remark 5.3. At this point we want to compare our approach with the one made by Dumortier, Roussarie and collaborators in [2, 6, 4, 13]. Let us explain succinctly the context in those papers using our notations. To this end let us assume that $\{X_\mu\}_{\mu\approx\mu_0}$ is an unfolding of a Hamiltonian vector field X_{μ_0} which has a center p with period annulus $\mathscr{P}$. Setting $U=\mathscr{P}\cup\{p\}$, we suppose additionally that ∂U is a hyperbolic 2-saddle cycle Γ for which at most one connection breaks when $\mu\approx\mu_0$. Similarly as we do in Figure 7 we take a transversal section Σ_1 in the unbroken connection and a transversal section Σ_2 in the other one, and we consider the difference map $\mathscr{D}(s;\mu)$ between the corresponding Dulac maps, which is defined on $(0,s_0)$. The limit cycles of X_μ that are Hausdorff close to $\Gamma=\partial U$ correspond to small isolated zeros of the displacement function $\mathscr{D}(s;\mu)$. If $f(s)$ is a smooth function on $(0,s_0)$ and $I\subset(0,s_0)$, we denote by $Z_I(f)$ (respectively, $Z_I^m(f)$) the number of zeros of f in I (respectively, counted with multiplicities). Then we have that

$$\mathrm{Cycl}_G((\partial U,X_{\mu_0}),X_\mu)=\mathrm{Cycl}((\Gamma,X_{\mu_0}),X_\mu)=\inf_{\varepsilon,\delta>0}\sup_{\mu\in B_\delta(\mu_0)}Z_{(0,\varepsilon)}\big(\mathscr{D}(\cdot;\mu)\big)=:\mathcal{Z}.$$

Moreover, if the boundary cyclicity of U from inside is finite then, by applying [10, Theorem 1] and the Weierstrass Preparation Theorem, it follows that

$$\underline{\mathrm{Cycl}}_G^U((\partial U,X_{\mu_0}),X_\mu)\leqslant\inf_{\varepsilon>0}\sup_{\xi(0)=\mu_0}Z^m_{(0,\varepsilon)}(M_\xi)=:\mathcal{M},$$

where the above supremum ranges over all the analytic arcs $\mu = \xi(\epsilon)$ with $\mathscr{D}\big(s;\xi(\epsilon)\big) \not\equiv 0$. Hence, by taking the Taylor's expansion at $\epsilon = 0$, we can write $\mathscr{D}\big(s;\xi(\epsilon)\big) = \epsilon^{k_\xi}(M_\xi(s) + O(\epsilon))$ where $M_\xi(s)$ is the first non-identically zero Melnikov function associated to the one-parameter unfolding $\{X_{\xi(\epsilon)}\}_{\epsilon\approx 0}$. (We point out that here we are using ε and ϵ.) Following this notation, $\mathcal{M} < \mathcal{Z}$ is a sufficient condition to have alien limit cycles bifurcating from ∂U because then the strict inequality (8) in Definition 1.8 holds. In contrast, our understanding is that the authors in [2, 6, 4, 13] define alien limit cycle bifurcation when $\mathcal{M}_1 < \mathcal{Z}$, where $\mathcal{M}_1$ is defined as $\mathcal{M}$ but taking the supremum only over all the analytic arcs ξ for which $k_\xi = 1$. On the other hand, Han and collaborators (see [27, 28, 30] and references therein) detect alien limit cycles according to Dumortier-Roussarie's definition by showing that there exist a family of arcs ξ with $k_\xi = 2$ for which the Melnikov function M_ξ has $m_2 > \mathcal{M}_1$ simple zeros arbitrarily close to $s = 0$. Then, by the implicit function theorem, they can assert that $\mathcal{M}_1 < m_2 \leqslant \mathcal{Z}$. □

A The asymptotic expansion of the Dulac map and related results

In order to prove Theorems A and B we will appeal to some previous results from [16, 17, 18] about the asymptotic expansion of the Dulac map. For reader's convenience we gather these results in Proposition A.4. To this end it is first necessary to introduce some new notation and definitions. For simplicity in the exposition, we use $\varpi \in \{\infty, \omega\}$ as a wild card in $\mathscr{C}^\varpi$ for the smooth class $\mathscr{C}^\infty$ and the analytic class $\mathscr{C}^\omega$.

Setting $\hat{\nu} := (\lambda, \nu) \in \hat{W} := (0, +\infty) \times W$ with W an open set of $\mathbb{R}^N$, we consider the family of vector fields $\{X_{\hat{\nu}}\}_{\hat{\nu}\in\hat{W}}$ with

$$X_{\hat{\nu}}(x_1, x_2) = x_1 P_1(x_1, x_2; \hat{\nu})\partial_{x_1} + x_2 P_2(x_1, x_2; \hat{\nu})\partial_{x_2} \tag{25}$$

where

- P_1 and P_2 belong to $\mathscr{C}^\varpi(\mathscr{U} \times \hat{W})$ for some open set $\mathscr{U}$ of $\mathbb{R}^2$ containing the origin,
- $P_1(x_1, 0; \hat{\nu}) > 0$ and $P_2(0, x_2; \hat{\nu}) < 0$ for all $(x_1, 0), (0, x_2) \in \mathscr{U}$ and $\hat{\nu} \in \hat{W}$,
- $\lambda = -\frac{P_2(0,0;\hat{\nu})}{P_1(0,0;\hat{\nu})}$.

Thus, for all $\hat{\nu} \in \hat{W}$, the origin is a hyperbolic saddle of $X_{\hat{\nu}}$ with the separatrices lying in the axis. We point out that here the hyperbolicity ratio of the saddle is an independent parameter, although in the applications we will have $\lambda = \lambda(\nu)$. The reason for this is that the hyperbolicity ratio turns out to be the ruling parameter in our results and, besides, having it uncoupled from the rest of parameters simplifies the notation in the statements. Moreover, for $i = 1, 2$, we consider a $\mathscr{C}^\varpi$ transverse section $\sigma_i \colon (-\varepsilon, \varepsilon) \times \hat{W} \longrightarrow \Sigma_i$ to $X_{\hat{\nu}}$ at $x_i = 0$ defined by

$$\sigma_i(s; \hat{\nu}) = \big(\sigma_{i1}(s; \hat{\nu}), \sigma_{i2}(s; \hat{\nu})\big)$$

such that $\sigma_1(0, \hat{\nu}) \in \{(0, x_2); x_2 > 0\}$ and $\sigma_2(0, \hat{\nu}) \in \{(x_1, 0); x_1 > 0\}$ for all $\hat{\nu} \in \hat{W}$. We denote the Dulac map of $X_{\hat{\nu}}$ from Σ_1 to Σ_2 by $D(\,\cdot\,; \hat{\nu})$, see Figure 9.

The asymptotic expansion of $D(s; \hat{\nu})$ at $s = 0$ consists of a remainder and a principal part. The principal part is given in a monomial scale that contains a deformation of the logarithm, the so-called Ecalle-Roussarie compensator, whereas the remainder has good flatness properties with respect to the parameters. We next give precise definitions of these key notions.

Definition A.1. The function defined for $s > 0$ and $\alpha \in \mathbb{R}$ by means of

$$\omega(s; \alpha) = \begin{cases} \frac{s^{-\alpha}-1}{\alpha} & \text{if } \alpha \neq 0, \\ -\log s & \text{if } \alpha = 0, \end{cases}$$

is called the *Ecalle-Roussarie compensator.* □

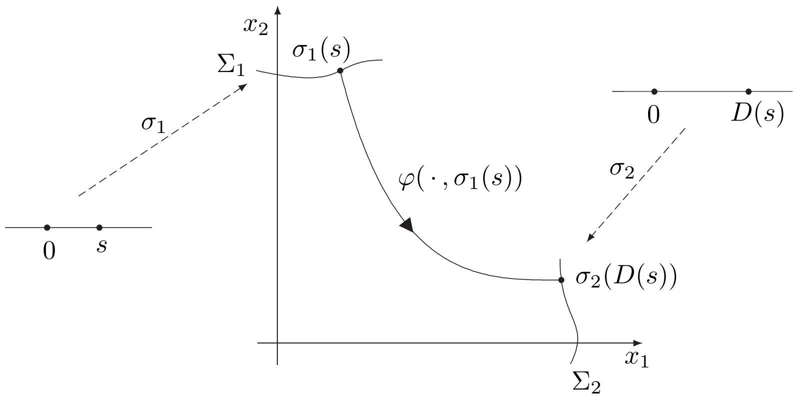

Figure 9: Definition of the Dulac map $D(\,\cdot\,;\hat{\nu})$, where $\varphi(t,p;\hat{\nu})$ is the solution of $X_{\hat{\nu}}$ passing through the point $p \in \mathscr{U}$ at time $t=0$.

Definition A.2. Consider $K \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and an open subset $U \subset \hat{W} \subset \mathbb{R}^{N+1}$. We say that a function $\psi(s;\hat{\nu})$ belongs to the class $\mathscr{C}^K_{s>0}(U)$, respectively $\mathscr{C}^K_{s=0}(U)$, if there exist an open neighbourhood Ω of

$$\{(s,\hat{\nu}) \in \mathbb{R}^{N+2}; s=0, \hat{\nu} \in U\} = \{0\} \times U$$

in $\mathbb{R}^{N+2}$ such that $(s,\hat{\nu}) \mapsto \psi(s;\hat{\nu})$ is $\mathscr{C}^K$ on $\Omega \cap \big((0,+\infty) \times U\big)$, respectively Ω. □

Definition A.3. Consider $K \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and an open subset $U \subset \hat{W} \subset \mathbb{R}^{N+1}$. Given $L \in \mathbb{R}$ and $\hat{\nu}_0 \in U$, we say that a function $\psi(s;\hat{\nu}) \in \mathscr{C}^K_{s>0}(U)$ is (L,K)*-flat with respect to* s *at* $\hat{\nu}_0$, and we write $\psi \in \mathcal{F}^K_L(\hat{\nu}_0)$, if for each $\ell = (\ell_0,\ldots,\ell_{N+1}) \in \mathbb{Z}^{N+2}_{\geq 0}$ with $|\ell| = \ell_0 + \ldots + \ell_{N+1} \leqslant K$ there exist a neighbourhood V of $\hat{\nu}_0$ and $C, s_0 > 0$ such that

$$\left| \frac{\partial^{|\ell|}\psi(s;\hat{\nu})}{\partial s^{\ell_0}\partial\hat{\nu}_1^{\ell_1}\cdots\partial\hat{\nu}_{N+1}^{\ell_{N+1}}} \right| \leqslant C s^{L-\ell_0} \text{ for all } s \in (0,s_0) \text{ and } \hat{\nu} \in V.$$

If W is a (not necessarily open) subset of U then define $\mathcal{F}^K_L(W) := \bigcap_{\hat{\nu}_0 \in W} \mathcal{F}^K_L(\hat{\nu}_0)$. □

Apart from the remainder and the monomial order, the most important ingredient for our purposes is the explicit expression of the coefficients in the asymptotic expansion. In order to give them we introduce next some additional notation, where for the sake of shortness the dependence on $\hat{\nu} = (\lambda,\nu)$ is omitted. We define the functions:

$$\begin{aligned} L_1(u) &:= \exp\int_0^u \left(\frac{P_1(0,z)}{P_2(0,z)} + \frac{1}{\lambda}\right)\frac{dz}{z} & L_2(u) &:= \exp\int_0^u \left(\frac{P_2(z,0)}{P_1(z,0)} + \lambda\right)\frac{dz}{z} \\ M_1(u) &:= L_1(u)\partial_1\!\left(\frac{P_1}{P_2}\right)(0,u) & M_2(u) &:= L_2(u)\partial_2\!\left(\frac{P_2}{P_1}\right)(u,0) \end{aligned} \tag{26}$$

On the other hand, for shortness as well, we use the compact notation σ_{ijk} for the kth derivative at $s=0$ of the jth component of $\sigma_i(s;\hat{\nu})$, i.e.,

$$\sigma_{ijk}(\hat{\nu}) := \partial_s^k \sigma_{ij}(0;\hat{\nu}).$$

Taking this notation into account we also introduce the following real values, where once again we omit the dependence on $\hat{\nu}$:

$$
\begin{aligned}
S_1 &:= \frac{\sigma_{112}}{2\sigma_{111}} - \frac{\sigma_{121}}{\sigma_{120}}\left(\frac{P_1}{P_2}\right)(0,\sigma_{120}) - \frac{\sigma_{111}}{L_1(\sigma_{120})}\hat{M}_1(1/\lambda,\sigma_{120})\\
S_2 &:= \frac{\sigma_{222}}{2\sigma_{221}} - \frac{\sigma_{211}}{\sigma_{210}}\left(\frac{P_2}{P_1}\right)(\sigma_{210},0) - \frac{\sigma_{221}}{L_2(\sigma_{210})}\hat{M}_2(\lambda,\sigma_{210})\\
S_3 &:= \frac{\sigma_{221}\sigma_{210}}{L_2(\sigma_{210})}M_2'(0).
\end{aligned}
\tag{27}
$$

Here $\hat{M}_i$ stands for a sort of incomplete Mellin transform of M_i that will be defined by Proposition A.5 below. The next proposition gathers the essential results in [18] that we shall need to prove the first main result in the present paper.

Proposition A.4. *Let $D(s;\hat{\nu})$ be the Dulac map of the hyperbolic saddle (25) from Σ_1 and Σ_2 and consider any $\lambda_0 > 0$. Then $D(s;\hat{\nu}) = \Delta_0(\hat{\nu})s^\lambda + \mathcal{F}_\ell^\infty(\{\lambda_0\}\times W)$ for any $\ell \in \big[\lambda_0, \min(2\lambda_0, \lambda_0+1)\big)$ where Δ_0 is a strictly positive $\mathscr{C}^\varpi$ function on $\hat{W}$ and*

$$
\Delta_0(\hat{\nu}) = \frac{\sigma_{111}^\lambda\sigma_{120}}{L_1^\lambda(\sigma_{120})}\frac{L_2(\sigma_{210})}{\sigma_{221}\sigma_{210}^\lambda}.
$$

Moreover,

(1) *If $\lambda_0 > 1$ then $D(s;\hat{\nu}) = \Delta_0(\hat{\nu})s^\lambda + \Delta_1(\hat{\nu})s^{\lambda+1} + \mathcal{F}_\ell^\infty(\{\lambda_0\}\times W)$ for any $\ell \in \big[\lambda_0+1, \min(\lambda_0+2, 2\lambda_0)\big)$ where Δ_1 is a $\mathscr{C}^\varpi$ function in a neighbourhood of $\{\lambda_0\}\times W$ and $\Delta_1(\hat{\nu}) = \Delta_0\lambda S_1$.*

(2) *If $\lambda_0 < 1$ then $D(s;\hat{\nu}) = \Delta_0(\hat{\nu})s^\lambda + \Delta_2(\hat{\nu})s^{2\lambda} + \mathcal{F}_\ell^\infty(\{\lambda_0\}\times W)$ for any $\ell \in \big[2\lambda_0, \min(3\lambda_0, \lambda_0+1)\big)$ where Δ_2 is a $\mathscr{C}^\varpi$ function in a neighbourhood of $\{\lambda_0\}\times W$ and $\Delta_2(\hat{\nu}) = -\Delta_0^2 S_2$.*

(3) *If $\lambda_0 = 1$ then $D(s;\hat{\nu}) = \Delta_0(\hat{\nu})s^\lambda + \Delta_3(\hat{\nu})s^{\lambda+1}\omega(s;1-\lambda) + \Delta_4(\hat{\nu})s^{\lambda+1} + \mathcal{F}_\ell^\infty(\{1\}\times W)$ for any $\ell \in [2,3)$ where Δ_3 and Δ_4 are $\mathscr{C}^\varpi$ functions in a neighbourhood of $\{1\}\times W$ and $\Delta_3(\hat{\nu})|_{\lambda=1} = -\Delta_0^2 S_3|_{\lambda=1}$.*

For the ease of the reader, let us explain regarding this result that the structure of the asymptotic expansion follows from [18, Theorem 4.1], whereas the properties (i.e., regularity and explicit expression) of the coefficients follow by applying Theorem A, Corollary B and Proposition 3.2 of the same paper. Furthermore, the flatness ℓ of the remainder can range in a certain interval depending on λ_0. The left endpoint of this interval is only given for completeness to guarantee that all the monomials in the principal part are relevant (i.e., they cannot be included in the remainder). The important information about the flatness is given by the right endpoint. A key tool in order to give a closed expression of the coefficients Δ_i is the use of a sort of incomplete Mellin transform, which is accurately defined in the next result. For a proof of this result the reader is referred to [18, Appendix B].

Proposition A.5. *Let us consider an open interval I of $\mathbb{R}$ containing $x=0$ and an open subset U of $\mathbb{R}^M$.*

(a) *Given $f(x;\upsilon) \in \mathscr{C}^\infty(I\times U)$, there exits a unique $\hat{f}(\alpha,x;\upsilon) \in \mathscr{C}^\infty((\mathbb{R}\setminus\mathbb{Z}_{\geqslant 0})\times I\times U)$ such that*

$$
x\partial_x\hat{f}(\alpha,x;\upsilon) - \alpha\hat{f}(\alpha,x;\upsilon) = f(x;\upsilon).
$$

(b) *If $x \in I\setminus\{0\}$ then $\partial_x\big(\hat{f}(\alpha,x;\upsilon)|x|^{-\alpha}\big) = f(x;\upsilon)\frac{|x|^{-\alpha}}{x}$ and, taking any $k\in\mathbb{Z}_{\geq 0}$ with $k>\alpha$,*

$$
\hat{f}(\alpha,x;\upsilon) = \sum_{i=0}^{k-1}\frac{\partial_x^i f(0;\upsilon)}{i!(i-\alpha)}x^i + |x|^\alpha\int_0^x\big(f(s;\upsilon) - T_0^{k-1}f(s;\upsilon)\big)\,|s|^{-\alpha}\frac{ds}{s},
$$

where $T_0^k f(x;\upsilon) = \sum_{i=0}^k\frac{1}{i!}\partial_x^i f(0;\upsilon)x^i$ is the k-th degree Taylor polynomial of $f(x;\upsilon)$ at $x=0$.

(c) *For each* $(i_0, x_0, \upsilon_0) \in \mathbb{Z}_{\geqslant 0} \times I \times W$ *the function* $(\alpha, x, \upsilon) \mapsto (i_0 - \alpha)\hat{f}(\alpha, x; \upsilon)$ *extends* $\mathscr{C}^\infty$ *at* (i_0, x_0, υ_0) *and, moreover, it tends to* $\frac{1}{i_0!}\partial_x^{i_0} f(0; \upsilon_0) x_0^{i_0}$ *as* $(\alpha, x, \upsilon) \to (i_0, x_0, \upsilon_0)$.

(d) *If* $f(x; \upsilon)$ *is analytic on* $I \times U$ *then* $\hat{f}(\alpha, x; \upsilon)$ *is analytic on* $(\mathbb{R} \setminus \mathbb{Z}_{\geqslant 0}) \times I \times U$. *Finally, for each* $(\alpha_0, x_0, \upsilon_0) \in \mathbb{Z}_{\geqslant 0} \times I \times U$ *the function* $(\alpha, x, \upsilon) \mapsto (\alpha_0 - \alpha)\hat{f}(\alpha, x; \upsilon)$ *extends analytically to* $(\alpha_0, x_0, \upsilon_0)$.

On account of this result for each $M_i(u; \hat{\nu})$ in (26) we have that $(\alpha, u; \hat{\nu}) \mapsto \hat{M}_i(\alpha, u; \hat{\nu})$ is a well defined meromorphic function with poles only at $\alpha \in \mathbb{Z}_{\geq 0}$. Accordingly, see (27), $\hat{M}_1(1/\lambda, \sigma_{120})$ and $\hat{M}_2(\lambda, \sigma_{210})$ are the values (depending on $\hat{\nu}$) that we obtain by taking $\hat{M}_1(\alpha, u; \hat{\nu})$ with $\alpha = 1/\lambda$ and $u = \sigma_{120}(\hat{\nu})$ and by taking $\hat{M}_2(\alpha, u; \hat{\nu})$ with $\alpha = \lambda$ and $u = \sigma_{210}(\hat{\nu})$, respectively.

The next result (see [17, Lemma 4.3]) is addressed to study the case in which the separatrices depicted in Figure 9 are not straight lines.

Lemma A.6. *Consider a* $\mathscr{C}^\infty$ *family* $\{X_\mu\}_{\mu \in \mathbb{R}^N}$ *of planar vector fields defined in some open set* W *of* $\mathbb{R}^2$. *Let us fix some* $\mu_0 \in \mathbb{R}^N$ *and assume that, for all* $\mu \approx \mu_0$, X_μ *has a hyperbolic saddle point at* $p_\mu \in W$ *with (global) stable and unstable separatrices* S_μ^+ *and* S_μ^-, *respectively. Consider two closed connected arcs* $\ell^\pm \subset S_{\mu_0}^\pm$, *having both an endpoint at* p_{μ_0}. *In case of a homoclinic connection (i.e.,* $S_{\mu_0}^+ = S_{\mu_0}^-$*) we require additionally that* $\ell^+ \cap \ell^- = \{p_{\mu_0}\}$. *Then there exists a neighbourhood* V *of* $(\ell^+ \cup \ell^-) \times \{\mu_0\}$ *in* $\mathbb{R}^2 \times \mathbb{R}^N$ *and a* $\mathscr{C}^\infty$ *diffeomorphism* $\Phi : V \to \Phi(V) \subset \mathbb{R}^2 \times \mathbb{R}^N$ *with* $\Phi(x, y, \mu) = (\phi_\mu(x, y), \mu)$ *such that*

$$\Phi((S_\mu^+ \times \{\mu\}) \cap V) \subset \{x = 0\} \times \{\mu\} \text{ and } \Phi((S_\mu^- \times \{\mu\}) \cap V) \subset \{y = 0\} \times \{\mu\}.$$

In other words, $(\phi_\mu)_\star(X_\mu) = \hat{X}_\mu$ *where* $\hat{X}_\mu(x, y) = xP(x, y; \mu)\partial_x + yQ(x, y; \mu)\partial_y$, *with* $P, Q \in \mathscr{C}^\infty(\Phi(V))$.

Next result gathers some general properties (see [16, Lemma A.3]) with regard to operations between functions in $\mathcal{F}_L^K(W)$ with $L \in \mathbb{R}$.

Lemma A.7. *Let* U *and* U' *be open sets of* $\mathbb{R}^N$ *and* $\mathbb{R}^{N'}$ *respectively and consider* $W \subset U$ *and* $W' \subset U'$. *Then the following holds:*

(a) $\mathcal{F}_L^K(W) \subset \mathcal{F}_L^K(\hat{W})$ *for any* $\hat{W} \subset W$ *and* $\bigcap_n \mathcal{F}_L^K(W_n) = \mathcal{F}_L^K\left(\bigcup_n W_n\right)$.

(b) $\mathcal{F}_L^K(W) \subset \mathcal{F}_L^K(W \times W')$.

(c) $\mathscr{C}^K(U) \subset \mathscr{C}_{s=0}^K(U) \subset \mathcal{F}_0^K(W)$.

(d) *If* $K \geqslant K'$ *and* $L \geqslant L'$ *then* $\mathcal{F}_L^K(W) \subset \mathcal{F}_{L'}^{K'}(W)$.

(e) $\mathcal{F}_L^K(W)$ *is closed under addition.*

(f) *If* $f \in \mathcal{F}_L^K(W)$ *and* $\nu \in \mathbb{Z}_{\geq 0}^{N+1}$ *with* $|\nu| \leqslant K$ *then* $\partial^\nu f \in \mathcal{F}_{L-\nu_0}^{K-|\nu|}(W)$.

(g) $\mathcal{F}_L^K(W) \cdot \mathcal{F}_{L'}^K(W) \subset \mathcal{F}_{L+L'}^K(W)$.

(h) *Assume that* $\phi \colon U' \longrightarrow U$ *is a* $\mathscr{C}^K$ *function with* $\phi(W') \subset W$ *and let us take* $g \in \mathcal{F}_{L'}^K(W')$ *with* $L' > 0$ *and verifying* $g(s; \eta) > 0$ *for all* $\eta \in W'$ *and* $s > 0$ *small enough. Consider also any* $f \in \mathcal{F}_L^K(W)$. *Then* $h(s; \eta) := f(g(s; \eta); \phi(\eta))$ *is a well-defined function that belongs to* $\mathcal{F}_{LL'}^K(W')$.

Remark A.8. From Definition A.3 it follows easily that if $\partial^\nu f \in \mathcal{F}_{L-\nu_0}^K(W)$ for all $\nu \in \mathbb{Z}_{\geq 0}^{N+1}$ with $|\nu| \leqslant 1$ then $f \in \mathcal{F}_L^{K+1}(W)$. This is a sort of converse for assertion (f) in Lemma A.7. $\square$

Lemma A.9. *Let us consider* $f(s; \mu) \in \mathcal{F}_\delta^\infty(\mu_0)$ *with* $\delta > 0$ *and define* $\psi(s, \mu) = \big(s(1 + f(s; \mu)), \mu\big)$ *for* $0 < s \ll 1$ *and* $\mu \approx \mu_0$. *Then* ψ *extends to a local* $\mathscr{C}^1$ *diffeomorphism on a neighbourhood of* $(0, \mu_0)$. *Moreover its inverse, for* $0 < s \ll 1$ *and* $\mu \approx \mu_0$, *writes as* $\psi^{-1}(s, \mu) = \big(s(1 + g(s; \mu)), \mu\big)$ *with* $g \in \mathcal{F}_\delta^\infty(\mu_0)$.

Proof. Since $f(s;\mu)\in\mathcal{F}_\delta^\infty(\mu_0)$ with $\delta>0$ then $sf(s;\mu)\in\mathcal{F}_{1+\delta}^\infty(\mu_0)$ extends to a $\mathscr{C}^1$ function on some neighbourhood of $(0,\mu_0)$ by [16, Lemma A.1]. Thus $F(s,u,\mu):=s(1+f(s;\mu))-u$ is $\mathscr{C}^1$ at $(0,0,\mu_0)$, $F(0,0,\mu_0)=0$ and $\partial_sF(0,0,\mu_0)=1$, and by applying the Implicit Function Theorem (see [11] for instance) there exists a unique $\mathscr{C}^1$ function $\sigma(u,\mu)$ on a neighbourhood $(-\varepsilon,\varepsilon)\times U$ of $(0,\mu_0)$ such that $\sigma(0,\mu_0)=0$ and $F(\sigma(u,\mu),u,\mu)\equiv 0$, i.e., $\sigma(u,\mu)(1+f(\sigma(u,\mu);\mu))=u$. Moreover the uniqueness implies that $\sigma(0,\mu)=0$ for all $\mu\in U$.

We claim that $\sigma\in\bigcap_{K=0}^\infty\mathcal{F}_1^K(\mu_0)=\mathcal{F}_1^\infty(\mu_0)$. The proof follows by induction on $K\in\mathbb{Z}_{\geq 0}$. Indeed, due to $\sigma(0,\mu)=0$ for all $\mu\in U$, we can write

$$\sigma(u,\mu)=u\int_0^1\partial_u\sigma(tu;\mu)dt\in u\mathscr{C}^0_{u=0}(U)\subset\mathcal{F}_1^0(\mu_0),$$

where the inclusion follows by (c) in Lemma A.7. Since $f\in\mathscr{C}^\infty_{s>0}(U)$, by applying the Implicit Function Theorem to the equality $F(s,u,\mu)=0$ at the points $(s,u,\mu)=(\sigma(u_\star,\mu_\star),u_\star,\mu_\star)$ with $(u_\star,\mu_\star)\in(0,\varepsilon)\times U$ and taking the uniqueness of σ into account, we deduce that $\sigma\in\mathscr{C}^\infty_{s>0}(U)$. Furthermore

$$\partial_u\sigma(u,\mu)=-\left(\frac{\partial_uF}{\partial_sF}\right)(\sigma(u,\mu),u,\mu)=\frac{1}{1+f_1(\sigma(u,\mu);\mu)}$$

and

$$\partial_{\mu_i}\sigma(u,\mu)=-\left(\frac{\partial_{\mu_i}F}{\partial_sF}\right)(\sigma(u,\mu),u,\mu)=-\frac{\sigma(u,\mu)\partial_{\mu_i}f(\sigma(u,\mu);\mu)}{1+f_1(\sigma(u,\mu);\mu)},$$

where $f_1:=f+s\partial_sf\in\mathcal{F}_\delta^\infty(\mu_0)$ by (f) in Lemma A.7. On account of these two expressions, and by applying Lemma A.7 once again, we can assert that if $\sigma\in\mathcal{F}_1^K(\mu_0)$ then $\partial_u\sigma\in\mathcal{F}_0^K(\mu_0)$ and $\partial_{\mu_i}\sigma\in\mathcal{F}_1^K(\mu_0)$, and consequently, see Remark A.8, $\sigma\in\mathcal{F}_1^{K+1}(\mu_0)$. Accordingly, since we already proved that $\sigma\in\mathcal{F}_1^0(\mu_0)$, we conclude that $\sigma\in\mathcal{F}_1^\infty(\mu_0)$ and $f(\sigma(u,\mu);\mu)\in\mathcal{F}_\delta^\infty(\mu_0)$ by induction. Hence

$$\sigma(u,\mu)=\frac{u}{1+f(\sigma(u,\mu);\mu)}=\frac{u}{1+\mathcal{F}_\delta^\infty(\mu_0)}=u(1+g(u,\mu))$$

with $g\in\mathcal{F}_\delta^\infty(\mu_0)$, thanks to Lemma A.7 again. This concludes the proof of the result. ∎

The following result is a kind of division theorem among the class of flat functions and its proof can be found in [19, Lemma 4.1].

Lemma A.10. *Let us fix $L\geqslant 0$ and $n\in\mathbb{N}$. If $f(s;\mu_1,\dots,\mu_n)\in\mathcal{F}_L^\infty(0_n)$ verifies that*

$$f(s;\mu_1,\dots,\mu_{k-1},0,\dots,0)\equiv 0,\ \text{for some } k\in\{1,2,\dots,n\},$$

then there exist $f_k,\dots,f_n\in\mathcal{F}_L^\infty(0_n)$ such that $f=\sum_{i=k}^n\mu_if_i$.

We give at this point the precise definition of independence of functions that we use in this paper and a subsequent result addressed to obtain lower bounds for the number of bifurcating zeros.

Definition A.11. Let W be a subset of $\mathbb{R}^N$ (not necessarily open) and consider the functions $g_i\colon W\longrightarrow\mathbb{R}$ for $i=1,2,\dots,k$. The *real variety* $V(g_1,g_2,\dots,g_k)\subset W$ is defined to be the set of $\mu\in W$ such that $g_i(\mu)=0$ for $i=1,2,\dots,k$. We say that $g_1,g_2,\dots,g_k$ are *independent* at $\mu_\star\in V(g_1,g_2,\dots,g_k)$ if the following conditions are satisfied:

(1) Every neighbourhood of $\mu_\star$ contains two points $\mu_1,\mu_2\in V(g_1,\dots,g_{k-1})$ such that $g_k(\mu_1)g_k(\mu_2)<0$ (if $k=1$ then we set $V(g_1,\dots,g_{k-1})=V(0)=W$ for this to hold).

(2) The varieties $V(g_1,\dots,g_i)$, $2\leqslant i\leqslant k-1$, are such that if $\mu_0\in V(g_1,\dots,g_i)$ and $g_{i+1}(\mu_0)\neq 0$, then every neighbourhood of μ_0 contains a point $\mu\in V(g_1,\dots,g_{i-1})$ such that $g_i(\mu)g_{i+1}(\mu_0)<0$.

(3) If $\mu_0\in V(g_1)$ and $g_2(\mu_0)\neq 0$, then every open neighbourhood of μ_0 contains a point $\mu\in W$ such that $g_1(\mu)g_2(\mu_0)<0$.

It is clear that if W is an open subset of $\mathbb{R}^N$ and $g_i\in\mathscr{C}^1(W)$ for $i=1,2,\dots,k$ then a sufficient condition for $g_1,g_2,\dots,g_k$ to be independent at $\mu_\star$ is that the gradients $\nabla g_1(\mu_\star),\nabla g_2(\mu_\star)\dots,\nabla g_k(\mu_\star)$ are linearly independent vectors of $\mathbb{R}^N$. □

Proposition A.12. *Let W be a subset of $\mathbb{R}^N$ (not necessarily open) and consider*

$$F(s;\mu)=\sum_{i=1}^{n}\delta_i(\mu)f_i(s;\mu)+f_{n+1}(s;\mu),$$

where $f_i\colon(0,\varepsilon)\times W\longrightarrow\mathbb{R}$ and $\delta_i\colon W\longrightarrow\mathbb{R}$ are continuous functions (with respect to the induced topology). If $\mu_\star\in V(\delta_1,\delta_2,\dots,\delta_n)\subset W$ satisfies

(*a*) *$F(s;\mu_\star)$ is not identically zero on $(0,\rho)$ for every $\rho\in(0,\varepsilon)$,*

(*b*) *$f_i(s;\mu)>0$, $1\leqslant i\leqslant n$, for all (s,μ) in a neighbourhood of $(0,\mu_\star)$ in $(0,\varepsilon)\times W$,*

(*c*) *$\lim_{s\to 0}\frac{f_{i+1}(s;\mu)}{f_i(s;\mu)}=0$, $1\leqslant i\leqslant n$, for every μ in a neighbourhood of $\mu_\star$ in W, and*

(*d*) *$\delta_1,\delta_2,\dots,\delta_n$ are independent at $\mu_\star$,*

then for every neighbourhood V of $\mu_\star$ in W and $\rho>0$ there exists $\mu_0\in V$ such that $F(s;\mu_0)$ has at least n different zeros inside the interval $(0,\rho)$.

Proof. Fix any $\rho>0$ and any neighbourhood U of $\mu_\star$ in W. Then, by the assumption (a), there exists $s_1\in(0,\rho)$ such that $F(s_1;\mu_\star)=f_{n+1}(s_1;\mu_\star)\neq 0$. Suppose for instance that $F(s_1;\mu_\star)>0$. Then, on account of (1) in Definition A.11, we can take $\mu_1\in U\cap V(\delta_1,\delta_2,\dots,\delta_{n-1})$ such that $\delta_n(\mu_1)<0$ and close enough to $\mu_\star$ so that, by continuity, $F(s_1;\mu_1)>0$. Observe that

$$F(s;\mu_1)=\delta_n(\mu_1)f_n(s;\mu_1)+f_{n+1}(s;\mu_1).$$

Thus, by (b) and (c), $\lim_{s\to 0}\frac{F(s;\mu_1)}{f_n(s;\mu_1)}=\delta_n(\mu_1)<0$ and we can take $s_2\in(0,s_1)$ such that $F(s_2;\mu_1)<0$. Next, thanks to (2) in Definition A.11, we can choose $\mu_2\in U\cap V(\delta_1,\delta_2,\dots,\delta_{n-2})$ with $\delta_{n-1}(\mu_2)>0$ and close enough to μ_1 so that $F(s_1;\mu_2)>0$ and $F(s_2;\mu_2)<0$. Note that

$$F(s;\mu_2)=\delta_{n-1}(\mu_2)f_{n-1}(s;\mu_2)+\delta_n(\mu_2)f_n(s;\mu_2)+f_{n+1}(s;\mu_2).$$

Consequently, by (b) and (c), $\lim_{s\to 0}\frac{F(s;\mu_2)}{f_{n-1}(s;\mu_2)}=\delta_{n-1}(\mu_2)>0$ and we can choose $s_3\in(0,s_2)$ such that $F(s_3;\mu_2)>0$. Next we take $\mu_3\in U\cap V(\delta_1,\delta_2,\dots,\delta_{n-3})$ with $\delta_{n-2}(\mu_3)<0$ and close enough to μ_2 so that $F(s_1;\mu_3)>0$, $F(s_2;\mu_3)<0$ and $F(s_3;\mu_3)>0$. We repeat this process $n-2$ times after which we find a parameter $\mu_{n+1}\in U$ and $0<s_{n+1}<s_n<\dots<s_2<s_1<\rho$, such that $(-1)^{i+1}F(s_i;\mu_{n+1})>0$ for all $i=1,2,\dots,n+1$. By applying Bolzano's theorem we can assert the existence of at least n different zeros of $F(\,\cdot\,;\mu_{n+1})$ inside the interval $(0,\rho)$. This concludes the proof of the result. ∎

B Deferred proofs

In this section, we collect the longest and most technical proofs.

B.1 Proof of Theorem 2.1

Proof of Theorem 2.1. We shall study first the Dulac map $D_+(\,\cdot\,;\mu)$ of X_μ from Σ_1 to Σ_2. For convenience we introduce auxiliary transverse sections Σ_1^η and Σ_2^η parametrized by $\sigma_1^\eta(s)=(\frac{\eta}{s},\frac{1}{s})$ and $\sigma_2^\eta(s)=(\eta,s)$ with $\eta\approx 0$, respectively. On the other hand, setting $\ell_\alpha:=x+\alpha(y+1)$, we perform the projective change of coordinates $(x_1,x_2)=\phi(x,y;\alpha):=(\frac{1}{\ell_\alpha},\frac{y}{\ell_\alpha})$ to the vector field X_μ, that recall is given by

$$\begin{cases} \dot{x}=yf(x,y;\mu)+g(x;\mu),\\ \dot{y}=yq(x,y;\mu). \end{cases}$$

In doing so we obtain that

$$\phi(\,\cdot\,;\alpha)_\star X_\mu=\frac{1}{x_1}\left(x_1\bar{P}_1(x_1,x_2;\mu,\alpha)\partial_{x_1}+x_2\bar{P}_2(x_1,x_2;\mu,\alpha)\partial_{x_2}\right),$$

where one can verify that

$$P_1(x_1,x_2;\mu):=\bar{P}_1(x_1,x_2;\mu,0)=x_2x_1^nf\left(\tfrac{1}{x_1},\tfrac{x_2}{x_1}\right)+x_1^{n+1}g\left(\tfrac{1}{x_1}\right) \tag{28}$$

and

$$P_2(x_1,x_2;\mu):=\bar{P}_2(x_1,x_2;\mu,0)=x_2x_1^nf\left(\tfrac{1}{x_1},\tfrac{x_2}{x_1}\right)+x_1^{n+1}g\left(\tfrac{1}{x_1}\right)-x_1^nq\left(\tfrac{1}{x_1},\tfrac{x_2}{x_1}\right). \tag{29}$$

Let us note at this point, see (2), that

$$\frac{P_2(x_1,x_2;\mu)}{P_1(x_1,x_2;\mu)}=1-\left.\frac{xq(x,y)}{yf(x,y)+g(x)}\right|_{(x,y)=\left(\frac{1}{x_1},\frac{x_2}{x_1}\right)}=K(x_1,x_2). \tag{30}$$

The origin $(x_1,x_2)=(0,0)$ is a hyperbolic saddle of $x_1\phi_\star(X_\mu;\alpha)$ with hyperbolicity ratio equal to

$$\lambda(\mu)=-K(0,0;\mu)=-1+\frac{q_n(1,0)}{g_{n+1}}.$$

By introducing α and η (that will make easier the forthcoming computations) we shall work in an extended parameter space $\bar{\mu}:=(\mu,\alpha,\eta)$ with the admissibility conditions $\Sigma_i^\eta\subset\{\ell_\alpha>0\}$ for $i=1,2$. Let $\bar{D}(\,\cdot\,;\mu,\alpha,\eta)$ be the Dulac map of $\bar{X}_{\bar{\mu}}:=x_1\phi(\,\cdot\,;\alpha)_\star X_\mu$ from Σ_1^η to Σ_2^η. The key point is that, by construction, $\bar{D}(\,\cdot\,;\mu,\alpha,\eta)$ does not depend on α and that $\bar{D}(\,\cdot\,;\mu,\alpha,0)=D_+(\,\cdot\,;\mu)$.

Let us fix any admissible α_0 and η_0. By applying Proposition A.4 to the analytic family of vector fields

$$\bar{X}_{\bar{\mu}}=x_1\bar{P}_1(x_1,x_2;\mu,\alpha)\partial_{x_1}+x_2\bar{P}_2(x_1,x_2;\mu,\alpha)\partial_{x_2}$$

at $\bar{\mu}_0=(\mu_0,\alpha_0,\eta_0)$ we can assert that

$$\bar{D}(s;\bar{\mu})=\bar{\Delta}_0(\bar{\mu})s^\lambda+\begin{cases}\bar{\Delta}_1(\bar{\mu})s^{\lambda+1}+\mathcal{F}^\infty_{\ell_1}(\bar{\mu}_0) & \text{if } \lambda_0>1,\\[1ex] \bar{\Delta}_2(\bar{\mu})s^{2\lambda}+\mathcal{F}^\infty_{\ell_2}(\bar{\mu}_0) & \text{if } \lambda_0<1,\\[1ex] \bar{\Delta}_3(\bar{\mu})s^{\lambda+1}\omega(s;1-\lambda)+\bar{\Delta}_4(\bar{\mu})s^{\lambda+1}+\mathcal{F}^\infty_{\ell_3}(\bar{\mu}_0) & \text{if } \lambda_0=1,\end{cases}$$

for any $\ell_1\in\big[\lambda_0+1,\min(2\lambda_0,\lambda_0+2)\big)$, $\ell_2\in\big[2\lambda_0,\min(3\lambda_0,\lambda_0+1)\big)$ and $\ell_3\in[2,3)$, respectively.

We remark that $\lambda_0=\lambda(\mu_0)=-K(0,0;\mu)$ because, although the new vector field $\bar{X}_{\bar{\mu}}$ depends on α, the hyperbolicity ratio of the saddle does not. We only need to compute the coefficients of the asymptotic development for $\eta=0$ and to this aim notice that

$$\Delta_i^+(\mu):=\bar{\Delta}_i(\mu,\alpha,0)=\lim_{\eta\to 0^+}\bar{\Delta}_i(\mu,\alpha,\eta)=\lim_{\eta\to 0^+}\bar{\Delta}_i(\mu,0,\eta),$$

where in the third equality we use that the coefficients do not depend on α. So it suffices to perform all the computations with $\alpha = 0$. The parametrisations of the auxiliary transverse sections Σ_1^η and Σ_2^η in coordinates (x_1, x_2) for $\alpha = 0$ are $\sigma_1(s) = (\frac{s}{\eta}, \frac{1}{\eta})$ and $\sigma_2(s) = (\frac{1}{\eta}, \frac{s}{\eta})$ respectively, so that $\sigma_{ijk} = \frac{1}{\eta}$ for $(i, j, k) \in \{(1, 1, 1), (1, 2, 0), (2, 1, 0), (2, 2, 1)\}$. Taking this into account, by applying Proposition A.4,

$$\bar{\Delta}_0(\mu, 0, \eta) = \exp\left(\int_0^{1/\eta} \left(\frac{P_2(z, 0)}{P_1(z, 0)} + \lambda - \lambda\frac{P_1(0, z)}{P_2(0, z)} - 1\right)\frac{dz}{z}\right),$$

where

$$\frac{P_2(z, 0)}{P_1(z, 0)} = 1 - \frac{q(1/z, 0)}{zg(1/z)} \quad \text{and} \quad \frac{P_1(0, z)}{P_2(0, z)} = 1 + \frac{q_n(1, z)}{zf_n(1, z) + g_{n+1} - q_n(1, z)} = 1 + \frac{q_n(1, z)}{\ell_{n+1}(1, z)}.$$

Consequently

$$\Delta_0^+(\mu) = \bar{\Delta}_0(\mu, \alpha, 0) = \lim_{\eta \to 0^+} \bar{\Delta}_0(\mu, 0, \eta) = \exp\left(-\int_0^{+\infty} \left(\frac{q(1/z, 0)}{zg(1/z)} + \lambda\frac{q_n(1, z)}{\ell_{n+1}(1, z)}\right)\frac{dz}{z}\right)$$

$$= \exp\left(-\int_0^{+\infty} \left(\frac{q(w, 0)}{g(w)} + \lambda\frac{q_n(w, 1)}{\ell_{n+1}(w, 1)}\right)dw\right). \tag{31}$$

In the third equality we apply the Dominated Convergence Theorem [25, Theorem 11.30] taking into account that the integrand does not grow faster than z^{-2} at infinity, which follows by the assumptions **H1** and **H2**. Moreover, in the last equality, we perform the change of coordinates $w = 1/z$ and take advantage of the homogeneity of the functions q_n and ℓ_{n+1}.

Next, we compute $\bar{\Delta}_2(\mu, \alpha, 0)$ under the assumption $\lambda_0 < 1$. By Proposition A.4, $\bar{\Delta}_2 = -(\bar{\Delta}_0)^2 S_2$ with

$$S_2 = \frac{\sigma_{222}}{2\sigma_{221}} - \frac{\sigma_{211}}{\sigma_{210}}\frac{P_2}{P_1}(\sigma_{210}, 0) - \frac{\sigma_{221}}{L_2(\sigma_{210})}\hat{M}_2(\lambda, \sigma_{210}) = -\frac{1/\eta}{L_2(1/\eta)}\hat{M}_2(\lambda, 1/\eta),$$

where, see (30) and (26), $M_2(u) = L_2(u)\partial_2 K(u, 0)$ with

$$L_2(u) = \exp\int_0^u (K(z, 0) + \lambda)\frac{dz}{z}$$

and we take $\sigma_2(s) = (\frac{1}{\eta}, \frac{s}{\eta})$ into account. To perform the limit of S_2 as $\eta \to 0^+$ we need to study the growth of the functions that are involved. With this aim observe that, since $\lambda(\mu) < 1$ for $\mu \approx \mu_0$, we can take $k = 1$ in (b) of Proposition A.5 to get

$$\hat{M}_2(\lambda, 1/\eta) = \frac{M_2(0)}{-\lambda} + \eta^{-\lambda}\int_0^{1/\eta} (M_2(u) - M_2(0))u^{-\lambda}\frac{du}{u}. \tag{32}$$

Setting $\tilde{f}(x_1, x_2) = x_1^n f\left(\frac{1}{x_1}, \frac{x_2}{x_1}\right)$, $\tilde{q}(x_1, x_2) = x_1^n q\left(\frac{1}{x_1}, \frac{x_2}{x_1}\right)$ and $\tilde{g}(x_1) = x_1^{n+1} g\left(\frac{1}{x_1}\right)$, from (28) and (29),

$$\partial_2 K(u, 0) = \partial_2\left(\frac{P_2}{P_1}\right)(u, 0) = \frac{\tilde{q}(u, 0)\tilde{f}(u, 0) - \partial_2\tilde{q}(u, 0)\tilde{g}(u)}{\tilde{g}(u)^2}.$$

Hence, using that $\deg(\tilde{g}) = n + 1$ due to $g(0) \neq 0$ (see **H1**) it follows that $\partial_2 K(u, 0)$ does not grow faster than u^{-2} at $u = +\infty$. We write this assertion as $\partial_2 K(u, 0) \prec u^{-2}$ and in what follows we shall use this notation for shortness. Since $(\lambda + 1)\int_1^{1/\eta} \frac{dz}{z} = -\log \eta^{1+\lambda}$, an easy computation yields

$$\log L_2(1/\eta) = \int_0^{1/\eta} (K(z, 0) + \lambda)\frac{dz}{z} = \int_0^{1/\eta} \left(1 - \frac{q(1/z, 0)}{zg(1/z)} + \lambda\right)\frac{dz}{z}$$

$$= \int_0^1 \left(1 - \frac{q(1/z, 0)}{zg(1/z)} + \lambda\right)\frac{dz}{z} - \int_\eta^1 \frac{q(w, 0)}{g(w)}dw - \log \eta^{\lambda+1}.$$

Accordingly, due to $g(0) \neq 0$ (see **H1**), setting

$$G_2^+ := \int_0^1 \left(\lambda + 1 - \frac{q(1/z,0)}{zg(1/z)} - \frac{zq(z,0)}{g(z)} \right) \frac{dz}{z},$$

the Dominated Convergence Theorem shows the validity of the limit

$$\lim_{\eta \to 0^+} \eta^{\lambda+1} L_2(1/\eta) = \exp(G_2^+). \tag{33}$$

In particular, $L_2(u) \prec u^{\lambda+1}$. Therefore $M_2(u) = L_2(u)\partial_2 K(u,0) \prec u^{\lambda-1}$. Hence, due to $\lambda < 1$, we can assert that $(M_2(u) - M_2(0))u^{-\lambda-1} \prec u^{-\lambda-1} \prec u^{-2}$. Accordingly, from (32),

$$\lim_{\eta \to 0^+} \eta^{\lambda} \hat{M}_2(\lambda, 1/\eta) = \int_0^{+\infty} \big(M_2(u) - M_2(0)\big) \frac{du}{u^{\lambda+1}}.$$

Finally, the combination of this with (33) yields

$$\begin{aligned} \Delta_2^+(\mu) = \bar{\Delta}_2(\mu,\alpha,0) = \lim_{\eta \to 0^+} \bar{\Delta}_2(\mu,0,\eta) = -\lim_{\eta \to 0^+} \big((\bar{\Delta}_0)^2 S_2\big)(\mu,0,\eta) \\ = (\Delta_0^+)^2 \exp(-G_2^+) \int_0^{+\infty} \big(M_2(u) - M_2(0)\big) \frac{du}{u^{\lambda+1}}. \end{aligned} \tag{34}$$

Our next task is to compute $\bar{\Delta}_1(\mu,\alpha,0)$ under the assumption $\lambda_0 > 1$, which is given by $\bar{\Delta}_1 = \lambda \bar{\Delta}_0 S_1$ thanks to the first assertion in Proposition A.4. Taking the derivatives of $\sigma_1(s) = (\frac{s}{\eta}, \frac{1}{\eta})$ at $s = 0$ into account we get that

$$S_1(\mu,0,\eta) = \frac{\sigma_{112}}{2\sigma_{111}} - \frac{\sigma_{121}}{\sigma_{120}} \frac{P_1}{P_2}(0,\sigma_{120}) - \frac{\sigma_{111}}{L_1(\sigma_{120})} \hat{M}_1(1/\lambda,\sigma_{120}) = -\frac{1}{\eta L_1(1/\eta)} \hat{M}_1(1/\lambda, 1/\eta),$$

where, see (30) and (26), $M_1(u) = L_1(u)\partial_1\big(\frac{1}{K}\big)(0,u)$ with

$$L_1(u) = \exp \int_0^u \left(\frac{1}{K(0,z)} + \frac{1}{\lambda} \right) \frac{dz}{z}.$$

Moreover

$$\partial_1 \left(\frac{1}{K} \right)(0,u) = \partial_1 \left(1 + \frac{\tilde{q}(x_1,x_2)}{x_2 \tilde{f}(x_1,x_2) + \tilde{g}(x_1) - \tilde{q}(x_1,x_2)} \right) \Big|_{(x_1,x_2)=(0,u)} \prec u^{-2}.$$

Here the assertion with regard to the growth at infinity is a consequence of $f_n(0,1) \neq 0$ (see **H2**), which implies that $\tilde{f}(0,u)$ has degree exactly n. On the other hand, by applying (*b*) in Proposition A.5 and taking $1/\lambda < 1$ into account, we get

$$\hat{M}_1(1/\lambda, 1/\eta) = -\lambda M_1(0) + \eta^{-1/\lambda} \int_0^{1/\eta} (M_1(u) - M_1(0)) u^{-1/\lambda} \frac{du}{u}. \tag{35}$$

Moreover

$$\begin{aligned} \log L_1(1/\eta) &= \int_0^{1/\eta} \left(\frac{1}{K(0,z)} + \frac{1}{\lambda} \right) \frac{dz}{z} = \int_0^{1/\eta} \left(\frac{q_n(1,z)}{\ell_{n+1}(1,z)} + 1 + \frac{1}{\lambda} \right) \frac{dz}{z} \\ &= \int_0^1 \left(\frac{q_n(1,z)}{\ell_{n+1}(1,z)} + 1 + \frac{1}{\lambda} \right) \frac{dz}{z} + \int_\eta^1 \frac{q_n(w,1)}{\ell_{n+1}(w,1)} dw - \left(1 + \frac{1}{\lambda} \right) \log \eta, \end{aligned}$$

where in the last equality we use the coordinate change $z = 1/w$. Consequently, by applying the Dominated Convergence Theorem using that $f_n(0,1) \neq 0$,

$$\lim_{\eta\to 0^+} \eta^{1+1/\lambda} L_1(1/\eta) = \exp(G_1^+), \tag{36}$$

where

$$G_1^+ := \int_0^1 \left(\frac{q_n(1,z)}{\ell_{n+1}(1,z)} + 1 + \frac{1}{\lambda} + \frac{z q_n(z,1)}{\ell_{n+1}(z,1)} \right) \frac{dz}{z}.$$

This implies in particular that $L_1(u) \prec u^{1+1/\lambda}$ and, accordingly, $M_1(u) \prec u^{-1+1/\lambda}$. The combination of this, together with (35) and (36), yields

$$\lim_{\eta\to 0^+} S_1(\mu,0,\eta) = -\lim_{\eta\to 0^+} \frac{\eta^{1/\lambda}\hat{M}_1(1/\lambda,1/\eta)}{\eta^{1+1/\lambda}L_1(1/\eta)} = -\exp(-G_1^+)\int_0^{+\infty} (M_1(u) - M_1(0))\frac{du}{u^{1+1/\lambda}}.$$

Therefore

$$\begin{aligned}\Delta_1^+(\mu) = \bar{\Delta}_1(\mu,\alpha,0) &= \big(\lambda\bar{\Delta}_0 S_1\big)(\mu,\alpha,0)\\ &= -\lambda\Delta_0^+\exp(-G_1^+)\int_0^{+\infty}\big(M_1(u)-M_1(0)\big)\frac{du}{u^{1+1/\lambda}}.\end{aligned} \tag{37}$$

Now we turn to the computation of the coefficient $\bar{\Delta}_3(\mu,\alpha,0)$ in case that $\lambda(\mu) = 1$. By the third assertion in Proposition A.4 we have that $\bar{\Delta}_3\big|_{\lambda=1} = -(\bar{\Delta}_0)^2 S_3\big|_{\lambda=1}$ with

$$S_3 = \frac{\sigma_{221}\sigma_{210}}{L_2(\sigma_{210})} M_2'(0).$$

Note that if $\lambda = 1$ then the quotient $\frac{\sigma_{221}\sigma_{210}}{L_2(\sigma_{210})} = \frac{1}{\eta^2 L_2(1/\eta)}$ tends to $\exp(-G_2^+)$ as $\eta \to 0^+$ thanks to (33), which is true for any $\lambda > 0$. Consequently, if $\lambda = 1$ then

$$\Delta_3^+(\mu) = \bar{\Delta}_3(\mu,\alpha,0) = \lim_{\eta\to 0^+}\bar{\Delta}_3(\mu,0,\eta) = -(\Delta_0^+)^2\exp(-G_2^+)M_2'(0). \tag{38}$$

So far we have proved that

$$D_+(s;\mu) = \Delta_0^+(\mu)s^\lambda + \begin{cases} \Delta_1^+(\mu)s^{\lambda+1} + \mathcal{F}_{\ell_1}^\infty(\mu_0) & \text{if } \lambda_0 > 1,\\ \Delta_2^+(\mu)s^{2\lambda} + \mathcal{F}_{\ell_2}^\infty(\mu_0) & \text{if } \lambda_0 < 1\\ \Delta_3^+(\mu)s^{\lambda+1}\omega(s;1-\lambda) + \Delta_4^+(\mu)s^{\lambda+1} + \mathcal{F}_{\ell_3}^\infty(\mu_0) & \text{if } \lambda_0 = 1.\end{cases}$$

We turn next to the study of the Dulac map $D_-(\,\cdot\,;\mu)$ of $-X_\mu$ from Σ_1 to Σ_2. To this aim the idea is to take advantage of the previous results for $D_+(\,\cdot\,;\mu)$ using the fact that $(x,y)\mapsto(-x,y)$ sends $-X_\mu$ to

$$\tilde{X}_\mu := \big(y\bar{f}(x,y;\mu) + \bar{g}(x;\mu)\big)\partial_x + y\bar{q}(x,y;\mu)\partial_y$$

with $\bar{f}(x,y) = f(-x,y)$, $\bar{g}(x) = g(-x)$ and $\bar{q}(x,y) = -q(-x,y)$. In particular, following the obvious notation one can check that $\bar{\ell}_{n+1}(x,y) = \ell_{n+1}(-x,y)$, together with

$$\bar{L}_i(u) = L_i(-u) \text{ and } \bar{M}_i(u) = -M_i(-u) \text{ for } i = 1,2 \tag{39}$$

is verified. By applying the above assertions to the Dulac map of $\tilde{X}_\mu$ from Σ_1 to Σ_2 we get that

$$D_-(s;\mu) = \Delta_0^-(\mu)s^\lambda + \begin{cases} \Delta_1^-(\mu)s^{\lambda+1} + \mathcal{F}_{\ell_1}^\infty(\mu_0) & \text{if } \lambda_0 > 1,\\ \Delta_2^-(\mu)s^{2\lambda} + \mathcal{F}_{\ell_2}^\infty(\mu_0) & \text{if } \lambda_0 < 1,\\ \Delta_3^-(\mu)s^{\lambda+1}\omega(s;1-\lambda) + \Delta_4^-(\mu)s^{\lambda+1} + \mathcal{F}_{\ell_3}^\infty(\mu_0) & \text{if } \lambda_0 = 1,\end{cases}$$

where each coefficient Δ_i^- is the counterpart for $\tilde{X}_\mu$ of the coefficient Δ_i^+ that we have obtained previously for X_μ. We can thus assert that

$$
\begin{aligned}
\mathscr{D}(s;\mu) &= D_+(s;\mu) - D_-(s;\mu) \\
&= \Delta_0(\mu)s^\lambda + \begin{cases} \Delta_1(\mu)s^{\lambda+1} + \mathcal{F}^\infty_{\ell_1}(\mu_0) & \text{if } \lambda_0 > 1, \\ \Delta_2(\mu)s^{2\lambda} + \mathcal{F}^\infty_{\ell_2}(\mu_0) & \text{if } \lambda_0 < 1, \\ \Delta_3(\mu)s^{\lambda+1}\omega(s;1-\lambda) + \Delta_4(\mu)s^{\lambda+1} + \mathcal{F}^\infty_{\ell_3}(\mu_0) & \text{if } \lambda_0 = 1, \end{cases}
\end{aligned}
$$

where $\Delta_i := \Delta_i^+ - \Delta_i^-$ for $i = 0,1,2,3,4$. Our next task is to compute each coefficient. Note that, from (31),

$$
\begin{aligned}
\Delta_0^-(\mu) &= \exp\left(\int_0^{+\infty}\left(\frac{q(-w,0)}{g(-w)} + \lambda\frac{q_n(-w,1)}{\ell_{n+1}(-w,1)}\right)dw\right) \\
&= \exp\left(\int_{-\infty}^{0}\left(\frac{q(z,0)}{g(z)} + \lambda\frac{q_n(z,1)}{\ell_{n+1}(z,1)}\right)dz\right).
\end{aligned}
$$

It is clear now that

$$
\log(\Delta_0^+) - \log(\Delta_0^-) = -\int_{-\infty}^{+\infty}\left(\frac{q(z,0)}{g(z)} + \lambda\frac{q_n(z,1)}{\ell_{n+1}(z,1)}\right)dz =: d_0
$$

On account of this, and the fact that $x \mapsto \log x$ is strictly increasing, the application of the mean value theorem shows that $\Delta_0 = \Delta_0^+ - \Delta_0^- = \kappa_0 d_0$ for some analytic function κ_0 with $\kappa_0(\mu_0) > 0$.

We turn next to the computation of Δ_2^-. To this end we again take advantage of the expression of Δ_2^+ thanks to the fact that $(x,y) \mapsto (-x,y)$ sends $-X_\mu$ to $\tilde{X}_\mu$. In doing so, recall (39), from (34) we get

$$
\Delta_2^- = -(\Delta_0^-)^2\exp(-G_2^-)\int_0^{+\infty}\big(M_2(-u) - M_2(0)\big)\frac{du}{u^{\lambda+1}}. \tag{40}
$$

where

$$
G_2^- := \int_0^1\left(\lambda + 1 + \frac{q(-1/z,0)}{zg(-1/z)} + \frac{zq(-z,0)}{g(-z)}\right)\frac{dz}{z}.
$$

In order to study $\Delta_2 = \Delta_2^+ - \Delta_2^-$ we first observe that

$$
\begin{aligned}
G_2^- - G_2^+ &= \int_0^1\left(\frac{q(1/z,0)}{zg(1/z)} + \frac{q(-1/z,0)}{zg(-1/z)}\right)\frac{dz}{z} + \int_0^1\left(\frac{q(z,0)}{g(z)} + \frac{q(-z,0)}{g(-z)}\right)dz \\
&= -\int_{+\infty}^1\left(\frac{q(u,0)}{g(u)} + \frac{q(-u,0)}{g(-u)}\right)du + \int_0^1\left(\frac{q(z,0)}{g(z)} + \frac{q(-z,0)}{g(-z)}\right)dz \\
&= \int_0^{+\infty}\left(\frac{q(z,0)}{g(z)} + \frac{q(-z,0)}{g(-z)}\right)dz =: G_2,
\end{aligned}
$$

where in the second equality we perform the change of coordinates $u = 1/z$. Then, from (34) and (40),

$$\begin{aligned}\Delta_2 = \Delta_2^+ - \Delta_2^- = \exp(-G_2^-)\Big((\Delta_0^-)^2 &\int_0^{+\infty} \big(M_2(-u) - M_2(0)\big)\frac{du}{u^{\lambda+1}} \\ &+(\Delta_0^+)^2\exp(G_2)\int_0^{+\infty} \big(M_2(u) - M_2(0)\big)\frac{du}{u^{\lambda+1}}\Big) \\ = \bar\kappa_2\Delta_0 + \exp(-G_2^-)(\Delta_0^-)^2 &\Big(\int_0^{+\infty} \big(M_2(-u) - M_2(0)\big)\frac{du}{u^{\lambda+1}} \\ &+\exp(G_2)\int_0^{+\infty} \big(M_2(u) - M_2(0)\big)\frac{du}{u^{\lambda+1}}\Big) \\ = \bar\kappa_2\Delta_0 + \kappa_2 F_2,&\end{aligned}$$

where in the second equality we use that $G_2^- - G_2^+ = G_2$, in the third one we plug $\Delta_0^+ = \Delta_0 + \Delta_0^-$ to get an analytic function $\bar\kappa_2 = \bar\kappa_2(\mu)$ multiplying Δ_0 and in the last one we set $\kappa_2 = \exp(-G_2^-)(\Delta_0^-)^2$. Accordingly $\Delta_2 = \kappa_2 F_2 + \bar\kappa_2\Delta_0$ with $\kappa_2(\mu_0) > 0$, so the assertion (2) in the statement is true.

In order to obtain the expression for $\Delta_1 = \Delta_1^+ - \Delta_1^-$ we follow the same strategy as before. First we take advantage of the expression of Δ_1^+ in (37) and the equalities in (39) to get that

$$\Delta_1^- = \lambda\Delta_0^-\exp(-G_1^-)\int_0^{+\infty} \big(M_1(-u) - M_1(0)\big)\frac{du}{u^{1+1/\lambda}}, \tag{41}$$

where

$$\begin{aligned}G_1^- &:= \int_0^1 \left(-\frac{q_n(-1,z)}{\ell_{n+1}(-1,z)} + 1 + \frac{1}{\lambda} - \frac{zq_n(-z,1)}{\ell_{n+1}(-z,1)}\right)\frac{dz}{z} \\ &= -\int_{-1}^0 \left(\frac{q_n(1,u)}{\ell_{n+1}(1,u)} + 1 + \frac{1}{\lambda} + \frac{uq_n(u,1)}{\ell_{n+1}(u,1)}\right)\frac{du}{u}.\end{aligned}$$

Here we use first the homogeneity of q_n and ℓ_{n+1} and then we perform the change of coordinates $u = -z$. Consequently

$$G_1^+ - G_1^- = \int_{-1}^1 \left(\frac{q_n(1,z)}{\ell_{n+1}(1,z)} + 1 + \frac{1}{\lambda} + \frac{zq_n(z,1)}{\ell_{n+1}(z,1)}\right)\frac{dz}{z} =: G_1$$

On account of this, the combination of (37) and (41) yields

$$\begin{aligned}\Delta_1 = \Delta_1^+ - \Delta_1^- = -\lambda\exp(-G_1^+)\Big(\Delta_0^+ &\int_0^{+\infty} \big(M_1(u) - M_1(0)\big)\frac{du}{u^{1+1/\lambda}} \\ &+\Delta_0^-\exp(G_1)\int_0^{+\infty} \big(M_1(-u) - M_1(0)\big)\frac{du}{u^{1+1/\lambda}}\Big) \\ = \bar\kappa_1\Delta_0 - \lambda\Delta_0^+\exp(-G_2^+) &\Big(\int_0^{+\infty} \big(M_1(u) - M_1(0)\big)\frac{du}{u^{1+1/\lambda}} \\ &+\exp(G_1)\int_0^{+\infty} \big(M_1(-u) - M_1(0)\big)\frac{du}{u^{1+1/\lambda}}\Big) \\ = \bar\kappa_1\Delta_0 + \kappa_1 F_1,&\end{aligned}$$

where in the second equality we use that $G_1^+ - G_1^- = G_1$, in the third one we replace Δ_0^+ by $\Delta_0 + \Delta_0^-$ to obtain a function $\bar\kappa_1$ multiplying Δ_0 and in the last one we set $\kappa_1 = \lambda\Delta_0^+\exp(-G_2^+)$. Therefore $\Delta_1 = \kappa_1 F_1 + \bar\kappa_1\Delta_0$ with $\kappa_1(\mu_0) > 0$. Since one can easily verify that $\bar\kappa_1$ is analytic at μ_0 with $\lambda(\mu_0) > 1$, this concludes the proof of assertion (1).

It only remains to compute $\Delta_3 = \Delta_3^+ - \Delta_3^-$ in case that $\lambda(\mu) = 1$. Exactly as before, since $(x, y) \mapsto (-x, y)$ sends $-X_\mu$ to $\tilde{X}_\mu$, from the expression of Δ_3^+ in (38) and taking (39) into account we get

$$\Delta_3^-\big|_{\lambda=1} = -(\Delta_0^-)^2 \exp(-G_2^-)\bar{M}_2'(0) = -(\Delta_0^-)^2 \exp(-G_2^-)M_2'(0).$$

Hence some straightforward computations show that

$$\begin{aligned}\Delta_3|_{\lambda=1} &= -\left((\Delta_0^+)^2 \exp(-G_2^+) - (\Delta_0^-)^2 \exp(-G_2^-)\right) M_2'(0)\\ &= -\left((\Delta_0 + \Delta_0^-)^2 \exp(-G_2 - G_2^-) - (\Delta_0^-)^2 \exp(-G_2^-)\right) M_2'(0)\\ &= \kappa_3 G_2 M_2'(0) + \bar{\kappa}_3 \Delta_0,\end{aligned}$$

where

$$\kappa_3 := (\Delta_0^-)^2 \exp(-G_2^-)\frac{1 - \exp(-G_2)}{G_2} \text{ and } \bar{\kappa}_3 := -(\Delta_0 + 2\Delta_0^-) \exp(-G_2^+)M_2'(0),$$

which are analytic functions at μ_0 and $\kappa_3(\mu_0) > 0$. Finally, due to $M_2(u) = L_2(u)\partial_2 K(u, 0)$ with

$$M_2(u) = L_2(u)\partial_2 K(u, 0) \text{ and } L_2(u) = \exp \int_0^u \left(K(z, 0) + \lambda\right)\frac{dz}{z},$$

one can easily show that $M_2'(0) = L_2'(0)\partial_2 K(0, 0) + L_2(0)\partial_{12}K(0, 0) = \partial_1 K(0, 0)\partial_2 K(0, 0) + \partial_{12}K(0, 0)$. We thus obtain that $\Delta_3|_{\lambda=1} = \kappa_3 F_3 + \bar{\kappa}_3 \Delta_0$ with $F_3 = G_2\big(\partial_1 K \partial_2 K + \partial_{12}K\big)(0, 0)$, as desired. This proves the validity of the third assertion in the statement and concludes the proof of the result. ■

B.2 Proof of Proposition 3.2

In this section we prove Proposition 3.2, which gives the asymptotic development of the difference map

$$\mathscr{D}_u(s; \mu) := D_+^u(s; \mu) - D_-^u(s; \mu),$$

see Figure 7, together with some properties of its coefficients. To this end we need first two auxiliary results.

Lemma B.1. *Fix any $\mu_0 = (a_0, b_0, \varepsilon_0, \varepsilon_1, \varepsilon_2)$ with $(a_0, b_0) \in (-2, 0) \times (0, 2)$ and $\varepsilon_i \approx 0$ for $i = 0, 1, 2$. Then*

$$D_\pm^u(s; \mu) = \delta_\pm + \Delta_0^\pm s^\lambda + \mathcal{F}_\ell^\infty(\mu_0), \text{ for any } \ell \in \big[\lambda_0, \min(2\lambda_0, \lambda_0 + 1)\big),$$

where λ, $\delta_\pm$ and $\Delta_0^\pm$ are $\mathscr{C}^\infty$ functions on $\mu \approx \mu_0$ and $\lambda_0 := \lambda(\mu_0) = -\frac{a_0+2}{a_0}$. Moreover, for $a_0 \neq -1$,

1. *If $a_0 > -1$ then $D_\pm^u(s; \mu) = \delta_\pm + \Delta_0^\pm s^\lambda + \Delta_1^\pm s^{\lambda+1} + \mathcal{F}_\ell^\infty(\mu_0)$ for any $\ell \in \big[\lambda_0 + 1, \min(2\lambda_0, \lambda_0 + 2)\big)$, where $\Delta_1^\pm$ is a $\mathscr{C}^\infty$ function on $\mu \approx \mu_0$.*
2. *If $a_0 < -1$ then $D_\pm^u(s; \mu) = \delta_\pm + \Delta_0^\pm s^\lambda + \Delta_2^\pm s^{2\lambda} + \mathcal{F}_\ell^\infty(\mu_0)$ for any $\ell \in \big[2\lambda_0, \min(3\lambda_0, \lambda_0 + 1)\big)$, where $\Delta_2^\pm$ is a $\mathscr{C}^\infty$ function on $\mu \approx \mu_0$.*

Proof. For the sake of simplicity in the exposition we omit the superscript in $D_\pm^u$. That being said, let us prove the result for the Dulac map $D_+(\,\cdot\,; \mu)$, the proof for $D_-(\,\cdot\,; \mu)$ follows verbatim. We denote the y-coordinate of the intersection point with $x = 0$ of the unstable separatrix of the saddle at s_1 by $\delta_+(\mu)$. Since the flow is smooth with respect to initial conditions and parameters, by applying the Implicit Function Theorem (see [11]) it follows that $\delta_+(\mu)$ is $\mathscr{C}^\infty$ in a neighbourhood of $\mu = \mu_0$. It is clear moreover that $\delta_+|_{\varepsilon_0=0} \equiv 0$. For convenience we change the parametrisation on Σ_2 by $\hat{s} \mapsto \big(0, \hat{s} + \delta_+(\mu)\big)$ for $\hat{s} > 0$ small enough and we denote by $\hat{D}_+(s; \mu)$ the Dulac map of X_μ from Σ_1 to Σ_2 with this new parametrisation in the arrival section. It is then clear that $D_+(s; \mu) = \delta_+(\mu) + \hat{D}_+(s; \mu)$ for $s > 0$. To study $\hat{D}_+(\,\cdot\,; \mu)$ we first compactify the vector field X_μ by using the projective coordinates $(u,v)=\phi_1(x,y):=(\frac{1}{x+y+1},\frac{y}{x+y+1})$. The key point here is that the trajectories of X_μ from Σ_1 to Σ_2 do not intersect $x+y+1=0$. In doing so we obtain an analytic family of vector fields which is orbitally equivalent to a polynomial one, say Y_μ, that has a finite hyperbolic saddle at the origin. By construction its stable separatrix is at $u=0$ for all μ, whereas its unstable one is at $v=0$ only when $\varepsilon_0=0$. In order to straighten both separatrices for all μ we apply Lemma A.6, that gives a $\mathscr{C}^\infty$ family of diffeomorphisms $\phi_2(u,v;\mu)$ such that the push-forward $(\phi_2)_\star(Y_\mu)$ writes as in (25) with $\varpi=\infty$. By construction, setting $\phi=\phi_2\circ\phi_1$, its Dulac map from $\phi(\Sigma_1)$ to $\phi(\Sigma_2)$, parametrised, respectively, by $\sigma_1(s;\mu)=\phi(0,1/s;\mu)$ and $\sigma_2(s;\mu)=\phi(0,s+\delta_+(\mu);\mu)$, is precisely $\hat{D}(s;\mu)$. Observe in this regard that the parametrisations of the transverse sections are $\mathscr{C}^\infty$. Accordingly, by applying Proposition A.4,

$$\hat{D}_+(s;\mu)=\Delta_0^+(\mu)s^\lambda+\begin{cases}\Delta_1^+(\mu)s^{\lambda+1}+\mathcal{F}^\infty_{\ell_1}(\mu_0) & \text{if } \lambda_0>1,\\[1ex] \Delta_2^+(\mu)s^{2\lambda}+\mathcal{F}^\infty_{\ell_2}(\mu_0) & \text{if } \lambda_0<1,\end{cases}$$

for any $\ell_2\in\big[2\lambda_0,\min(3\lambda_0,\lambda_0+1)\big)$ and $\ell_1\in\big[\lambda_0+1,\min(2\lambda_0,\lambda_0+2)\big)$. Here $\lambda=\lambda(\mu)$ is the hyperbolicity ratio of the saddle of X_μ at s_1 and $\lambda_0=\lambda(\mu_0)=-\frac{a_0+2}{a_0}$. Moreover the coefficient Δ_0^+ is $\mathscr{C}^\infty$ at μ_0 and, on the other hand, the coefficient Δ_1^+ (respectively, Δ_2^+) is $\mathscr{C}^\infty$ at μ_0 provided that $\lambda_0>1$ (respectively, $\lambda_0<1$). On account of $D_+(s;\mu)=\delta_+(\mu)+\hat{D}_+(s;\mu)$ this concludes the proof of the result. ■

Lemma B.2. *$\partial_{\varepsilon_0}\big(\delta_+-\delta_-\big)(\mu)<0$ for all $\mu=(a,b,0,0,0)$ with $a\in(-2,0)\setminus\{-1\}$ and $b\in(0,2)$.*

Proof. The differential form associated to system (13) is given by

$$\Omega:=\big(2xy-\varepsilon_0\big)dx+\big(\tfrac{b-2}{4}+\varepsilon_1x+(1-b)y+ax^2+\varepsilon_2xy+by^2\big)dy.$$

We know on the other hand that

$$H(x,y):=y(x^2+\ell y^2+my+n)^{\frac{1}{a}},\tag{42}$$

with $\ell=\frac{b}{a+2}$, $m=-\frac{b-1}{a+1}$ and $n=\frac{b-2}{4a}$, is a first integral of (13) for $\varepsilon_0=\varepsilon_1=\varepsilon_2=0$. We observe in this regard that

$$a\frac{dH}{H}=a\frac{dy}{y}+\frac{2xdx+(2\ell y+m)dy}{x^2+\ell y^2+my+n},$$

which yields

$$ay^{1-a}H^{a-1}dH=2xy\,dx+\big(an+(a+1)my+ax^2+(a+2)\ell y^2\big)dy=\Omega|_{\varepsilon_1=\varepsilon_2=0}+\varepsilon_0dx,$$

where in the second equality we use the expression of ℓ, m and n in terms of a and b. This shows that $\Omega|_{\varepsilon_1=\varepsilon_2=0}$ is proportional to $\Omega_0:=dH-\frac{\varepsilon_0}{a}H^{1-a}y^{a-1}dx$. On account of this, if we take any $\mu_0=(a,b,\varepsilon_0,0,0)$ and denote by Γ_{s,ε_0} the oriented arc of orbit of X_{μ_0} that joins the points $\big(0,D_+^u(s;\mu_0)\big)$ and $\big(0,D_-^u(s;\mu_0)\big)$ then we have that

$$0=\int_{\Gamma_{s,\varepsilon_0}}\Omega_0=H\big(0,D_+^u(s;\mu_0)\big)-H\big(0,D_-^u(s;\mu_0)\big)-\frac{\varepsilon_0}{a}\int_{\Gamma_{s,\varepsilon_0}}H(x,y)^{1-a}y^{a-1}dx,$$

where $D_\pm^u(s;\mu)$ is Dulac map in Lemma B.1. Consequently

$$H\big(0,D_+^u(s;\mu_0)\big)-H\big(0,D_-^u(s;\mu_0)\big)=\frac{\varepsilon_0}{a}\int_{\Gamma_{s,\varepsilon_0}}H(x,y)^{1-a}y^{a-1}dx \text{ for all } \varepsilon_0\approx0.$$

The derivative of this expression with respect to ε_0 evaluated at $\bar{\mu}_0:=(a,b,0,0,0)$ yields

$$\partial_yH\big(0,D_+^u(s;\bar{\mu}_0)\big)\partial_{\varepsilon_0}D_+^u(s;\bar{\mu}_0)-\partial_yH\big(0,D_-^u(s;\bar{\mu}_0)\big)\partial_{\varepsilon_0}D_-^u(s;\bar{\mu}_0)=\frac{1}{a}\int_{\Gamma_{s,0}}H(x,y)^{1-a}y^{a-1}dx.$$

Our next goal will be to make $s \to 0^+$ in this equality. With this aim in view note that, by the first assertion in Lemma B.1, $D^u_\pm(s;\mu) = \delta_\pm(\mu) + \mathcal{F}^\infty_\rho(\mu_0)$ for any $\rho > 0$ small enough. Consequently, since $\delta_\pm(\bar\mu_0) = 0$, we get that

$$\lim_{s\to 0^+} \partial_y H\big(0, D^u_\pm(s;\bar\mu_0)\big)\partial_{\varepsilon_0} D^u_\pm(s;\bar\mu_0) = \partial_y H(0,0)\partial_{\varepsilon_0}\delta_\pm(\bar\mu_0) = n^{1/a}\partial_{\varepsilon_0}\delta_\pm(\bar\mu_0),$$

where in the first equality we use the good properties of the remainder with respect to the derivation of the parameters, see Definition A.3, and in the second one the expression in (42). Therefore

$$an^{1/a}\big(\partial_{\varepsilon_0}\delta_+(\bar\mu_0) - \partial_{\varepsilon_0}\delta_-(\bar\mu_0)\big) = \lim_{s\to 0^+}\int_{\Gamma_{s,0}} H(x,y)^{1-a}y^{a-1}dx. \tag{43}$$

Note at this point that $\Gamma_{s,0}$ is a periodic orbit of $X_{\bar\mu_0}$. Thus it is contained inside the level set $H(x,y) = h$ where $h = h(s)$ verifies

$$h = H(0,1/s) = s^{-1-2/a}(\ell + ms + ns^2)^{\frac{1}{a}}.$$

Here we use (42) once again and that the parametrization of Σ_1 is given by $s \mapsto (0,1/s)$. Since $a \in (-2,0)$ by assumption, this shows that $\lim_{s\to 0^+} h(s) = 0$. Accordingly, if we denote by γ_h the periodic orbit of $X_{\bar\mu_0}$ inside the level curve $H = h$, from (43) we get that

$$an^{1/a}\partial_{\varepsilon_0}\big(\delta_+ - \delta_-\big)(\bar\mu_0) = \lim_{h\to 0} h^{1-a}\int_{\gamma_h} y^{a-1}dx.$$

It is clear then that the result will follow once we prove that the above limit exists and is different from zero. To this end, setting $\gamma_h^+ := \gamma_h \cap \{x \geqslant 0\}$, we first observe that

$$\int_{\gamma_h} y^{a-1}dx = 2\int_{\gamma_h^+} y^{a-1}dx$$

since $X_{\bar\mu_0}$ is symmetric with respect to $x = 0$. To compute this Abelian integral we perform the projective change of coordinates $(u,v) = (\frac{1}{x}, \frac{y}{x})$ and in these new variables, see (42), we have that

$$\gamma_h^+ \subset \{\hat H(u,v) = h^a\}, \text{ where } \hat H(u,v) := u^{-a-2}v^a(1 + \ell v^2 + muv + nu^2).$$

A computation shows that

$$\frac{\partial_u \hat H(u,v)}{\partial_v \hat H(u,v)} = -\frac{u}{v}\frac{(u-2v)\big((b-2)u - 2bv\big) + 4a}{(u-2v)\big((b-2)u - 2bv\big) + 4(a+2)}, \tag{44}$$

which gives, up to a unity, the expression of the partial derivatives of $\hat H$. Then, taking $(a,b) \in (-2,0)\times(0,2)$ into account, it follows that $\partial_v \hat H(u,v) \neq 0$ on $0 < u \leqslant 2v$ and $\partial_u \hat H(u,v) \neq 0$ on $0 < 2v \leqslant u$. Observe also that, for each $h > 0$, the arc γ_h^+ has exactly one intersection point with the straight line $u = 2v$ because $\hat H(u,u) = h^a$ if, and only if, $u = \pm c(h)$ where $c(h) := (2^{a+2}h^a - (\ell + 2m + 4n))^{-1/2}$. Therefore, by applying (twice) the Implicit Function Theorem to $\hat H(u,v) = h^a$ we can split γ_h^+ as

$$\gamma_h^+ = \big\{u = u(v;h), v \in [c(h), +\infty)\big\} \cup \big\{v = v(u;h), u \in [c(h), +\infty)\big\}.$$

Accordingly, from (42) once again,

$$\begin{aligned}\lim_{h\to 0} h^{1-a}\int_{\gamma_h^+} y^{a-1}dx &= \lim_{h\to 0}\int_{\gamma_h^+}(x^2 + \ell y^2 + my + n)^{\frac{1}{a}-1}dx \\ &= \lim_{h\to 0}\Bigg(-\int_{+\infty}^{c(h)} (1 + \ell v^2 + muv + nu^2)^{\frac{1-a}{a}}\Big|_{v=v(u;h)} u^{-\frac{2}{a}}du \\ &\qquad - \int_{c(h)}^{+\infty}(1 + \ell v^2 + mvu + nu^2)^{\frac{1-a}{a}}u^{-\frac{2}{a}}\Big|_{u=u(v;h)}\partial_v u(v;h)dv\Bigg).\end{aligned}$$

In order to make this limit let us first observe that $\lim_{h\to 0} c(h) = 0$ due to $a < 0$. On the other hand, $\lim_{h\to 0} u(v;h) = 0$, uniformly in v, and $\lim_{h\to 0} v(u;h) = 0$, uniformly in u, because the oval γ_h tends to the polycycle (in Hausdorff sense) as $h \to 0$. Furthermore, due to

$$\partial_v u(v;h) = \frac{du}{dv} = -\left.\frac{\partial_v \hat{H}(u,v)}{\partial_u \hat{H}(u,v)}\right|_{u=u(v;h)},$$

from the expression in (44) we deduce that $|\partial_v u(v;h)|$ is uniformly bounded since $0 < u(v;h) \leqslant 2v$ for any $v \in [c(h), +\infty)$. Taking these facts into account, together with the assumption $a \in (-2,0)$, by applying the Dominated Convergence Theorem we conclude that

$$\lim_{h\to 0} h^{1-a} \int_{\gamma_h^+} y^{a-1} dx = \int_0^{+\infty} (1+nu^2)^{\frac{1-a}{a}} u^{-\frac{2}{a}} du =: p \in \mathbb{R}_{>0}.$$

Hence $\partial_{\varepsilon_0}\big(\delta_+ - \delta_-\big)(\bar{\mu}_0) = \frac{2pn^{-1/a}}{a} < 0$ and this finishes the proof of the result. ∎

Proof of Proposition 3.2. The three assertions with regard to structure of the asymptotic development follow from Lemma B.1 setting $\Delta_i^u := \Delta_i^+ - \Delta_i^-$ for $i = 0,1,2$ and $\delta_u := \delta_+ - \delta_-$ because then

$$\mathscr{D}_u(s;\mu) = D_+(s;\mu) - D_-(s;\mu) = \delta_u(\mu) + \Delta_0^u(\mu)s^\lambda + \begin{cases} \Delta_1^u(\mu)s^{\lambda+1} + \mathcal{F}_{\ell_1}^\infty(\mu_0) & \text{if } a_0 > -1, \\ \Delta_2^u(\mu)s^{2\lambda} + \mathcal{F}_{\ell_2}^\infty(\mu_0) & \text{if } a_0 < -1, \end{cases} \tag{45}$$

for any $\ell_2 \in \big[2\lambda_0, \min(3\lambda_0, \lambda_0+1)\big)$ and $\ell_1 \in \big[\lambda_0+1, \min(2\lambda_0, \lambda_0+2)\big)$. Since we will deal with the "upper case" only, for simplicity in the exposition we shall omit any subscript and superscript u from now on.

It is clear that $\mathscr{D}(s;\mu_0) \equiv 0$ because X_μ is inside the center variety when $\mu = \mu_0$. On the other hand, by Lemma B.2, $\partial_{\varepsilon_0}\delta(\mu_0) < 0$. Note also that the straight line $y = 0$ is invariant in case that $\varepsilon_0 = 0$. Hence $\delta(\mu)|_{\varepsilon_0=0} \equiv 0$ by definition and, consequently, $\partial_{\varepsilon_1}\delta(\mu_0) = \partial_{\varepsilon_2}\delta(\mu_0) = 0$. That being stablished, our main task is to compute the partial derivatives $\partial_{\varepsilon_1}\Delta_k$ and $\partial_{\varepsilon_2}\Delta_k$ evaluated at $\mu_0 = (a_0, b_0, 0, 0, 0)$ for each $k = 0,1,2$. To this end the key point is that we can perform the computations setting $\varepsilon_0 = 0$ and that in this case X_μ is a D-system, more concretely, with $f(x,y) = 1 - b + \varepsilon_2 x + by$, $g(x) = \frac{b-2}{4} + \varepsilon_1 x + ax^2$, $q(x,y) = -2x$ and $n = 1$, so that

$$\ell_2(x,y) = (a+2)x^2 + \varepsilon_2 xy + by^2.$$

Let us remark that it is only for $\varepsilon_0 = 0$ that X_μ becomes a D-system. Thus, for the sake of consistency we shall denote $\bar{\mu} = (a, b, \varepsilon_1, \varepsilon_2)$ and $\bar{\mu}_0 = (a, b, 0, 0)$. That being said, following the notation in Theorem 2.1, that we stress it is addressed to D-systems, from (2) we have

$$K(x_1, x_2; \bar{\mu}) = 1 - \left.\frac{xq(x,y)}{yf(x,y) + g(x)}\right|_{(x,y)=\left(\frac{1}{x_1}, \frac{x_2}{x_1}\right)} = 1 + \frac{2}{a + \varepsilon_1 x_1 + \varepsilon_2 x_2 + \frac{b-2}{4}x_1^2 + (1-b)x_1 x_2 + bx_2^2}.$$

Hence $\lambda(\bar{\mu}) = -\frac{a+2}{a}$. From (6) we get that

$$\begin{aligned} d_0(\bar{\mu}) &= 2\int_{-\infty}^{+\infty} \left(\frac{z}{\frac{b-2}{4} + \varepsilon_1 z + az^2} + \lambda \frac{z}{(a+2)z^2 + \varepsilon_2 z + b} \right) dz \\ &= \frac{2\pi}{a} \left(\frac{\varepsilon_1}{\sqrt{(b-2)a - \varepsilon_1^2}} + \frac{\varepsilon_2}{\sqrt{4b(a+2) - \varepsilon_2^2}} \right). \end{aligned}$$

On account of this one can verify that $d_0(\bar\mu) = -\rho_0(\bar\mu)\Big(2\frac{\sqrt{b(a+2)}}{\sqrt{a(b-2)}}\,\varepsilon_1+\varepsilon_2\Big)$ where ρ_0 is a smooth function with $\rho_0(\bar\mu_0)>0$ since $a_0\in(-2,0)$. Hence, from (45) and applying Theorem 2.1,

$$\Delta_0(\mu)|_{\varepsilon_0=0} = -\kappa_{01}(\bar\mu)\left(2\frac{\sqrt{b(a+2)}}{\sqrt{a(b-2)}}\,\varepsilon_1+\varepsilon_2\right) \text{ with } \kappa_{01}(\bar\mu_0)>0.$$

Consequently, there exists a smooth function $\rho_1=\rho_1(\mu)$ such that

$$\Delta_0(\mu) = -\kappa_{01}(\bar\mu)\left(2\frac{\sqrt{b(a+2)}}{\sqrt{a(b-2)}}\,\varepsilon_1+\varepsilon_2\right)+\varepsilon_0\rho_1(\mu) = -\kappa_{01}(\bar\mu)\left(2\frac{\sqrt{b(a+2)}}{\sqrt{a(b-2)}}\,\varepsilon_1+\varepsilon_2\right)+\kappa_{02}(\mu)\delta(\mu),$$

where in the second equality we use that we can write $\delta(\mu)=\varepsilon_0\rho_2(\mu)$ with $\rho_2(\mu_0)\neq 0$ due to $\delta(\mu)|_{\varepsilon_0=0}\equiv 0$ and $\partial_{\varepsilon_0}\delta(\mu_0)\neq 0$. Since $\kappa_{01}(\bar\mu)$ is a smooth function on μ, this proves the assertion with regard to $\Delta_0(\mu)$.

Let us assume now that $a_0\in(-2,-1)$ and turn to the study of Δ_2. This, on account of Theorem 2.1, leads to the computation of F_2. According to (4) its expression is given by

$$F_2(\bar\mu) = \int_0^{+\infty}\Big(M_2(-z)-M_2(0)+\exp(G_2)\big(M_2(z)-M_2(0)\big)\Big)\frac{dz}{z^{1+\lambda}} \tag{46}$$

where $M_2(u)=L_2(u)\partial_2K(u,0)$ with $L_2(u) := \exp\big(\int_0^u \big(K(z,0)+\lambda\big)\frac{dz}{z}\big)$. After some lengthy computations we obtain that

$$L_2(u) = \Big(1+\frac{\varepsilon_1}{a}u+\eta_2u^2\Big)^{-\frac{1}{a}}\,B_2(u),$$

where $\eta_2 := \frac{b-2}{4a}>0$ for all $(a,b)\in(-2,0)\times(0,2)$ and

$$B_2(u) := \exp\left(\frac{-2\varepsilon_1}{a\sqrt{a(b-2)-\varepsilon_1^2}}\arctan\left(\frac{\sqrt{a\,(b-2)-\varepsilon_1^2}\,u}{\varepsilon_1u+2a}\right)\right).$$

The explicit computation of $F_2(\bar\mu)$ for arbitrary $\bar\mu$ requires a primitive of $u\mapsto (M_2(u)-M_2(0))\,u^{-1-\lambda}$, which is not feasible because $M_2(u)=L_2(u)\partial_2K(u,0)$ where

$$\partial_2K(u,0) = \frac{2}{a^2}\,\frac{(b-1)u-\varepsilon_2}{\big(1+\frac{\varepsilon_1}{a}u+\eta_2u^2\big)^2}.$$

To bypass this problem the strategy is to compute only the first order Taylor's expansion of this function at $(\varepsilon_1,\varepsilon_2)=(0,0)$. In doing so we get

$$\begin{aligned}M_2(u)-M_2(0) = {}&\frac{2(b-1)}{a^2}\,u(1+\eta_2u^2)^{-2-\frac{1}{a}}\\ &-\frac{2(b-1)}{a^4\sqrt{\eta_2}}\,u(1+\eta_2u^2)^{-3-\frac{1}{a}}\big((1+\eta_2u^2)\arctan(\sqrt{\eta_2}u)+\sqrt{\eta_2}(1+2a)u\big)\varepsilon_1\\ &-\frac{2}{a^2}\big((1+\eta_2u^2)^{-2-\frac{1}{a}}-1\big)\varepsilon_2+\mathrm{o}(\|(\varepsilon_1,\varepsilon_2)\|).\end{aligned}$$

Thus, on account of the parity of each coefficient with respect to u, if we write

$$\int_0^{+\infty}\big(M_2(\pm u)-M_2(0)\big)\frac{du}{u^{1+\lambda}} = m_0^{\pm}+m_1^{\pm}\varepsilon_1+m_2^{\pm}\varepsilon_2+\mathrm{o}(\|(\varepsilon_1,\varepsilon_2)\|) \tag{47}$$

then it turns out that $m_0^-=-m_0^+$, $m_1^-=m_1^+$ and $m_2^-=m_2^+$. Of course to obtain the above equality we must prove that the higher order terms can also be neglected after integration. To show this let us note first that, as a matter of fact, the higher order terms do not depend on ε_2 because $M_2(u;\bar{\mu})$ is linear in this parameter. Therefore to get $m_0^{\pm}$ we need a result to pass the limit $\varepsilon_1 \to 0$ under the integral sign, and to get $m_1^{\pm}$ a similar result for the derivation with respect to ε_1. With this aim we appeal to the results in [31, §17.2] about improper integrals depending on a parameter. More concretely, Proposition 2, which is a sort of Weierstrass test for the uniform convergence of an improper integral depending on a parameter, and Proposition 6, that gives sufficient conditions for the differentiation of an improper integral with respect to a parameter. To this end the key points are that, on one hand, $\lambda = \lambda(\bar{\mu}) = -\frac{a+2}{a} \in (0,1)$ for $\bar{\mu} \approx \bar{\mu}_0$ due to $a_0 \in (-2,-1)$ and, on the other hand, that $B_2(u;\bar{\mu})$ and $\partial_{\varepsilon_1} B_2(u;\bar{\mu})$ are bounded for $u \in (0,+\infty)$ and $\varepsilon_1 \approx 0$ by a constant. That being said, some computations show that

$$m_0^+ = \frac{2(b-1)}{a^2} \int_0^{+\infty} (1+\eta_2 u^2)^{-2-\frac{1}{a}} u^{\frac{2}{a}+1} du = \frac{(b-1)\eta_2{}^{-1-\frac{1}{a}}}{a(a+1)}$$

and

$$m_2^+ = -\frac{2}{a^2} \int_0^{+\infty} \left((1+\eta_2 u^2)^{-2-\frac{1}{a}} - 1\right) u^{\frac{2}{a}} du = -\frac{\sqrt{\pi}}{2a^2} \frac{\Gamma\left(\frac{a+2}{2a}\right)}{\Gamma\left(\frac{2a+1}{a}\right)} \eta_2{}^{-\frac{a+2}{2a}}.$$

One can readily check in particular that $m_2^+ > 0$ for all $(a,b) \in (-2,-1) \times (0,2)$. Computing m_1^+ is a little more involved. In this case

$$\begin{aligned}
-\frac{a^4}{2(b-1)} m_1^+ &= \frac{1}{\sqrt{\eta_2}} \int_0^{+\infty} (1+\eta_2 u^2)^{-2-\frac{1}{a}} \arctan(\sqrt{\eta_2} u) u^{1+\frac{2}{a}} du + (1+2a) \int_0^{+\infty} (1+\eta_2 u^2)^{-3-\frac{1}{a}} u^{2+\frac{2}{a}} du \\
&= \frac{a\pi \eta_2{}^{-\frac{3a+2}{2a}}}{4(1+a)} - \frac{a\sqrt{\pi}}{4(a+1)} \frac{\Gamma\left(\frac{3a+2}{2a}\right)}{\Gamma\left(\frac{2a+1}{a}\right)} \eta_2{}^{-\frac{3a+2}{2a}} + (1+2a) \frac{\sqrt{\pi}}{4} \frac{\Gamma\left(\frac{3a+2}{2a}\right)}{\Gamma\left(\frac{3a+1}{a}\right)} \eta_2{}^{-\frac{3a+2}{2a}},
\end{aligned}$$

and after some simplifications we get that

$$m_1^+ = -\frac{\sqrt{\pi}(b-1)}{2a^2(a+1)} \left(\frac{\Gamma\left(\frac{3a+2}{2a}\right)}{\Gamma\left(\frac{2a+1}{a}\right)} + \frac{\sqrt{\pi}}{a} \right) \eta_2{}^{-\frac{3a+2}{2a}}.$$

On the other hand,

$$G_2 = \int_0^{+\infty} \left(\frac{q(u,0)}{g(u)} + \frac{q(-u,0)}{g(-u)} \right) du = -\frac{2\pi\varepsilon_1}{a\sqrt{(b-2)a - \varepsilon_1^2}},$$

so that $\exp(G_2) = 1 + \frac{2\pi}{\sqrt{a^3(b-2)}}\varepsilon_1 + \mathrm{o}(\varepsilon_1)$ due to $a<0$. Accordingly the substitution of (47) in (46) yields

$$\begin{aligned}
F_2 &= m_0^- + m_1^- \varepsilon_1 + m_2^- \varepsilon_2 + \left(1 + \frac{2\pi}{\sqrt{a^3(b-2)}}\varepsilon_1\right) \left(m_0^+ + m_1^+ \varepsilon_1 + m_2^+ \varepsilon_2\right) + \mathrm{o}(\|(\varepsilon_1,\varepsilon_2)\|) \\
&= m_0^- + m_0^+ + \left(m_1^- + m_1^+ + \frac{2\pi}{\sqrt{a^3(b-2)}} m_0^+\right) \varepsilon_1 + \left(m_2^- + m_2^+\right) \varepsilon_2 + \mathrm{o}(\|(\varepsilon_1,\varepsilon_2)\|) \\
&= 2\left(m_1^+ + \frac{\pi}{\sqrt{a^3(b-2)}} m_0^+\right) \varepsilon_1 + 2m_2^+ \varepsilon_2 + \mathrm{o}(\|(\varepsilon_1,\varepsilon_2)\|)
\end{aligned}$$

and let us note that

$$m_1^+ + \frac{\pi}{\sqrt{a^3(b-2)}} m_0^+ = -\frac{\sqrt{\pi}}{2a^2} \frac{b-1}{a+1} \frac{\Gamma\left(\frac{3a+2}{2a}\right)}{\Gamma\left(\frac{2a+1}{a}\right)} \eta_2{}^{-\frac{3a+2}{2a}}.$$

Hence $m_1^+ + \frac{\pi}{\sqrt{a^3(b-2)}} m_0^+ = \frac{2(a+2)(b-1)}{(a+1)(b-2)} m_2^+$, so that

$$F_2(\bar{\mu}) = \rho_3(\bar{\mu}) \left(\frac{2(a+2)(b-1)}{(a+1)(b-2)} \varepsilon_1 + \varepsilon_2 + \mathrm{o}(\|(\varepsilon_1, \varepsilon_2)\|) \right) \text{ with } \rho_3(\bar{\mu}_0) > 0.$$

Finally, from (45) and the last assertion in (2) of Theorem 2.1 we get that

$$\Delta_2(\mu)|_{\varepsilon_0=0} = \kappa_{21}(\bar{\mu}) \left(\frac{2(a+2)(b-1)}{(a+1)(b-2)} \varepsilon_1 + \varepsilon_2 + \mathrm{o}(\|(\varepsilon_1, \varepsilon_2)\|) \right) + \kappa_{22}(\bar{\mu})\, \Delta_0(\mu)|_{\varepsilon_0=0} \text{ with } \kappa_{21}(\bar{\mu}_0) > 0.$$

Let us stress here that κ_{21} and κ_{22} are smooth functions in a neighbourhood of $\bar{\mu}_0$ provided that $a_0 \in (-2,-1)$. Consequently, since $\delta(\mu)|_{\varepsilon_0=0} \equiv 0$ and $\partial_{\varepsilon_0}\delta(\mu_0) \neq 0$, we get that

$$\Delta_2(\mu) = \kappa_{21}(\bar{\mu}) \left(\frac{2(a+2)(b-1)}{(a+1)(b-2)} \varepsilon_1 + \varepsilon_2 + \mathrm{o}(\|(\varepsilon_1, \varepsilon_2)\|) \right) + \kappa_{22}(\bar{\mu})\Delta_0(\mu) + \kappa_{23}(\mu)\delta(\mu)$$

for some smooth function κ_{23} and this proves the assertion in (2).

So far we have studied the coefficient Δ_2 assuming $a_0 \in (-2,-1)$, i.e., $\lambda_0 < 1$. Our next task is to do the same for the coefficient Δ_1 assuming $a_0 \in (-1,0)$, i.e., $\lambda_0 > 1$. In this case, see (3), we have to compute

$$F_1(\mu) = -\int_0^{+\infty} \Big(M_1(z) - M_1(0) + \exp(G_1)\big(M_1(-z) - M_1(0)\big) \Big) \frac{dz}{z^{1+1/\lambda}}, \tag{48}$$

where $M_1(u) = L_1(u)\partial_1\big(\frac{1}{K}\big)(0,u)$ with $L_1(u) := \exp\left(\int_0^u \left(\frac{1}{K(0,z)} + \frac{1}{\lambda} \right) \frac{dz}{z} \right)$. In doing so exactly as before we obtain that

$$L_1(u) = \left(1 + \frac{\varepsilon_2}{a+2} u + \eta_1 u^2 \right)^{\frac{1}{a+2}} B_1(u),$$

where $\eta_1 := \frac{b}{a+2} > 0$ for all $(a,b) \in (-2,0) \times (0,2)$ and

$$B_1(u) := \exp\left(\frac{2\varepsilon_2}{(a+2)\sqrt{4b(a+2) - \varepsilon_2^2}} \left(\arctan\left(\frac{\sqrt{4b(a+2) - \epsilon_1^2}\, u}{\epsilon_1 u + 2(a+2)} \right) \right) \right).$$

Since one can also verify that

$$\partial_1 \left(\frac{1}{K} \right)(0,u) = \frac{2}{(a+2)^2} \frac{(1-b)u + \varepsilon_1}{\left(1 + \frac{\varepsilon_2}{a+2}u + \eta_1 u^2\right)^2},$$

it turns out that the function $M_1(u) = L_1(u)\partial_1\big(\frac{1}{K}\big)(0,u)$ is linear in ε_1. That being said, some computations show that

$$\begin{aligned} M_1(u) - M_1(0) &= \frac{2(1-b)}{(a+2)^2} u (1+\eta_1 u^2)^{-\frac{2a+3}{a+2}} \\ &+ \frac{2}{(a+2)^2} \left((1+\eta_1 u^2)^{-\frac{2a+3}{a+2}} - 1 \right) \varepsilon_1 \\ &+ \frac{2(1-b)u}{(a+2)^3 \sqrt{b(a+2)}} (1+\eta_1 u^2)^{-\frac{2a+3}{a+2}} \left(\arctan(\sqrt{\eta_1} u) - \frac{(2a+3)\sqrt{\eta_1} u}{1+\eta_1 u^2} \right) \varepsilon_2 + \mathrm{o}(\|(\varepsilon_1, \varepsilon_2)\|). \end{aligned}$$

Following the obvious notation, if we write

$$\int_0^{+\infty} \big(M_1(\pm u) - M_1(0) \big) u^{-1-1/\lambda} du = n_0^\pm + n_1^\pm \varepsilon_1 + n_2^\pm \varepsilon_2 + \mathrm{o}(\|(\varepsilon_1, \varepsilon_2)\|),$$

then $n_0^- = -n_0^+$, $n_1^- = n_1^+$ and $n_2^- = n_2^+$ due to the parity of each coefficient with respect to u. Here we follow exactly the same strategy as before, by using the results from [31, §17.2] about improper integrals depending on a parameter, to show that the higher order terms can be neglected after integration. Moreover

$$G_1 = \int_{-1}^{1} \left(\frac{q_n(1,z)}{\ell_{n+1}(1,z)} + 1 + \frac{1}{\lambda} + \frac{z q_n(z,1)}{\ell_{n+1}(z,1)} \right) \frac{dz}{z} = \frac{2\pi\varepsilon_2}{(a+2)\sqrt{4b(a+2)-\varepsilon_2^2}},$$

so that $\exp(G_1) = 1 + \frac{\pi}{(a+2)\sqrt{b(a+2)}}\varepsilon_2 + \mathrm{o}(\varepsilon_2)$. Accordingly, from (48) we can assert that

$$\begin{aligned} F_1 &= -(n_0^+ + n_1^+\varepsilon_1 + n_2^+\varepsilon_2) - \left(1 + \frac{\pi}{\sqrt{b(a+2)^3}}\varepsilon_2\right)\left(n_0^- + n_1^-\varepsilon_1 + n_2^-\varepsilon_2\right) + \mathrm{o}(\|(\varepsilon_1,\varepsilon_2)\|) \\ &= -n_0^- - n_0^+ - \left(n_1^- + n_1^+\right)\varepsilon_1 - \left(n_2^- + n_2^+ + \frac{\pi}{\sqrt{b(a+2)^3}} n_0^-\right)\varepsilon_2 + \mathrm{o}(\|(\varepsilon_1,\varepsilon_2)\|) \\ &= -2n_1^+\varepsilon_1 - \left(2n_2^+ - \frac{\pi}{\sqrt{b(a+2)^3}} n_0^+\right)\varepsilon_2 + \mathrm{o}(\|(\varepsilon_1,\varepsilon_2)\|). \end{aligned}$$

In order to compute this coefficients let us note that

$$n_0^+ = \frac{2(1-b)}{(a+2)^2}\int_0^{+\infty} (1+\eta_1 u^2)^{-\frac{2a+3}{a+2}} \frac{du}{u^{1/\lambda}} = \frac{1-b}{(a+1)(a+2)}\eta_1^{-\frac{a+1}{a+2}}$$

and

$$n_1^+ = \frac{2}{(a+2)^2}\int_0^{+\infty} \left((1+\eta_1 u^2)^{-\frac{2a+3}{a+2}} - 1\right) \frac{du}{u^{1+1/\lambda}} = \frac{\sqrt{\pi}}{2(a+2)^2}\frac{\Gamma\left(\frac{a}{2(a+2)}\right)}{\Gamma\left(\frac{2a+3}{a+2}\right)}\eta_1^{-\frac{a}{2(a+2)}}.$$

The computations of n_2^+ is a little more involved. In this case

$$\begin{aligned} \frac{(a+2)^3\sqrt{b(a+2)}}{2(1-b)} n_2^+ &= \int_0^{+\infty} (1+\eta_1 u^2)^{-\frac{2a+3}{a+2}} \arctan(\sqrt{\eta_1}u) u^{-1/\lambda} du \\ &\quad - (2a+3)\sqrt{\eta_1}\int_0^{+\infty} (1+\eta_1 u^2)^{-\frac{3a+5}{a+2}} u^{1-1/\lambda} du \\ &= \frac{\sqrt{\pi}(a+2)^2}{4(a+1)}\left(\frac{\Gamma\left(\frac{3a+4}{2(a+2)}\right)}{\Gamma\left(\frac{2a+3}{a+2}\right)} - \frac{\sqrt{\pi}}{a+2}\right)\eta_1^{-\frac{a+1}{a+2}}, \end{aligned}$$

where to obtain the expression of the first integral we perform integration by parts. From here some additional computations show that

$$2n_2^+ - \frac{\pi}{\sqrt{b(a+2)^3}} n_0^+ = \frac{\sqrt{\pi}(b-1)\eta_1^{-\frac{a+1}{a+2}}}{(a+1)\sqrt{b(a+2)^3}}\frac{\Gamma\left(\frac{3a+4}{2(a+2)}\right)}{\Gamma\left(\frac{2a+3}{a+2}\right)}$$

and, on account of this,

$$\frac{2n_2^+ - \frac{\pi}{\sqrt{b(a+2)^3}} n_0^+}{2n_1^+} = \frac{a(b-1)}{2(a+1)b}.$$

Since $n_1^+ < 0$ for all $a \in (-1,0)$ and $b \in (0,2)$, we have that $F_1(\bar{\mu}) = \rho_4(\bar{\mu})\left(\varepsilon_1 + \frac{a(b-1)}{2(a+1)b}\varepsilon_2 + \mathrm{o}(\|(\varepsilon_1,\varepsilon_2)\|)\right)$ with $\rho_4(\bar{\mu}_0) > 0$. Accordingly, the combination of (45) and the last assertion in (1) of Theorem 2.1 yields

$$\Delta_1(\mu)|_{\varepsilon_0=0} = \kappa_{11}(\bar{\mu})\left(\varepsilon_1 + \frac{a(b-1)}{2(a+1)b}\varepsilon_2 + \mathrm{o}(\|(\varepsilon_1,\varepsilon_2)\|)\right) + \kappa_{12}(\bar{\mu})\,\Delta_0(\mu)|_{\varepsilon_0=0}$$

with $\kappa_{11}(\bar{\mu}_0) > 0$. Finally, once again thanks to $\partial_{\varepsilon_0}\delta(\mu_0) \neq 0$, we can write

$$\Delta_1(\mu) = \kappa_{11}(\bar{\mu})\left(\varepsilon_1 + \frac{a(b-1)}{2(a+1)b}\varepsilon_2 + \mathrm{o}(\|(\varepsilon_1,\varepsilon_2)\|)\right) + \kappa_{12}(\bar{\mu})\Delta_0(\mu) + \kappa_{13}(\mu)\delta(\mu)$$

for some smooth function κ_{13} in a neighbourhood of μ_0. This proves the last assertion in (1) and completes the proof of the result. ■

References

[1] J.C. Artés, F. Dumortier and J. Llibre, "Qualitative theory of planar differential systems", Universitext, Springer-Verlag, Berlin, 2006.

[2] M. Caubergh, F. Dumortier and R. Roussarie, *Alien limit cycles in rigid unfoldings of a Hamiltonian 2-saddle cycle,* Commun. Pure Appl. Anal. **6** (2007) 1–21.

[3] B. Coll, C. Li, R. Prohens, *Quadratic perturbations of a class of quadratic reversible systems with two centers*, Discrete and Continuous Dynamical Systems, **24** (2009) 699–729.

[4] B. Coll, F. Dumortier, and R. Prohens, *Alien limit cycles in Liénard equations,* J. Differerential Equations **254** (2013) 1582–1600.

[5] F. Dumortier, A. Guzmán and C. Rousseau, *Finite cyclicity of elementary graphics surrounding a focus or center in quadratic systems,* Qual. Theory Dyn. Syst. **3** (2002) 123–154.

[6] F. Dumortier and R. Roussarie, *Abelian integrals and limit cycles*, J. Differential Equations, **224** (2006) 296–313.

[7] F. Dumortier, R. Roussarie and C. Rousseau, *Hilbert's 16th problem for quadratic vector fields,* J. Differential Equations **110** (1994) 86–133.

[8] J.-P. Françoise and L. Gavrilov, *Perturbation theory of the quadratic Lotka-Volterra double center*, Commun. Contemp. Math. **24** (2022), no. 5, paper no. 2150064, 38 pp.

[9] A. Gasull, V. Mañosa and F. Mañosas, *Stability of certain planar unbounded polycycles,* J. Math. Anal. Appl. **269** (2002) 332–351.

[10] L. Gavrilov, *Cyclicity of period annuli and principalization of Bautin ideals*, Ergod. Th. & Dynam. Sys. **28** (2008) 1497–1507.

[11] Ed. Goursat, *Sur la théorie des fontions implicites,* Bulletin de la S.M.F. **31** (1903) 184–192.

[12] I.D. Iliev, *Perturbations of quadratic centers*, Bull. Sci. Math. **122** (1998) 107–161.

[13] S. Luca, F. Dumortier, M. Caubergh and R. Roussarie, *Detecting alien limit cycles near a Hamiltonian 2-saddle cycle,* Discrete Contin. Dyn. Syst. **4** (2009) 723–781.

[14] C. Liu, *The cyclicity of period annuli of a class of quadratic reversible systems with two centers*, J. Differential Equations **252** (2012) 5260–5273.

[15] A. Mourtada, *Action de derivations irreductibles sur les algebres quasi-regulieres d'Hilbert*, preprint (2009), arXiv:0912.1560v1.

[16] D. Marín and J. Villadelprat, *Asymptotic expansion of the Dulac map and time for unfoldings of hyperbolic saddles: local setting,* J. Differential Equations **269** (2020) 8425–8467.

[17] D. Marín and J. Villadelprat, *Asymptotic expansion of the Dulac map and time for unfoldings of hyperbolic saddles: general setting*, J. Differential Equations, **275** (2021) 684–732.

[18] D. Marín and J. Villadelprat, *Asymptotic expansion of the Dulac map and time for unfoldings of hyperbolic saddles: coefficient properties*, preprint.

[19] D. Marín and J. Villadelprat, *The criticality of reversible quadratic centers at the outer boundary of its period annulus*, J. Differential Equations, **332** (2022), 123–201.

[20] J.R. Munkres, "Topology: a first course", Prentice-Hall, Inc., Englewood Cliffs, NJ, 1975

[21] L. Peng, Z. Feng and C. Liu, *Quadratic perturbations of a quadratic reversible Lotka-Volterra system with two centers*, Discrete and Continuous Dynamical Systems **34** (2014) 4807–4826.

[22] R. Roussarie, "Bifurcations of planar vector fields and Hilbert's sixteenth problem" [2013] reprint of the 1998 edition. Modern Birkhäuser Classics. Birkhäuser/Springer, Basel, 1998.

[23] R. Roussarie and C. Rousseau, *Finite cyclicity of nilpotent graphics of pp-type surrounding a center,* Bull. Belg. Math. Soc. Simon Stevin **15** (2008) 889–920.

[24] C. Rousseau, *Normal forms, bifurcations and finiteness properties of vector fields*, NATO Sci. Ser. II Math. Phys. Chem., **137**, Kluwer Academic Publishers, Dordrecht (2004) 431–470.

[25] W. Rudin, "Real and complex analysis" McGraw-Hill Book Co., New York-Toronto, Ont.-London 1966.

[26] D. S. Shafer and A. Zegeling, *Bifurcation of limit cycles from quadratic centers*, J. Differential Equations **122** (1995), 48–70.

[27] L. Sheng and M. Han, *Bifurcation of limit cycles from a compound loop with five saddles,* J. Appl. Anal. Comput. **9** (2019) 2482–2495.

[28] L. Sheng, M. Han and Y. Tian, *On the number of limit cycles bifurcating from a compound polycycle,* Int. J. Bifur. Chaos Appl. Sci. Eng. **30** 2050099 (2020).

[29] G. Swirszcz, *Cyclicity of Infinite Contour around Certain Reversible Quadratic Center*, J. Differential Equations, **154** (1999) 239–266.

[30] J. Yang, Y. Xiong and M. Han, *Limit cycle bifurcations near a 2-polycycle or double 2-polycycle of planar systems,* **95** (2014) Nonlinear Anal. 756–773.

[31] V. A. Zorich, "Mathematical analysis II" Translated from the 2002 fourth Russian edition by Roger Cooke. Universitext. Springer-Verlag, Berlin, 2004.