THE EASIEST POLYNOMIAL DIFFERENTIAL SYSTEMS IN $\mathbb{R}^3$ HAVING AN INVARIANT CYLINDER

JAUME LLIBRE[1] AND LEONARDO P. SERANTOLA[2]

ABSTRACT. This paper answers the following two questions: *What are the easiest polynomial differential systems in* $\mathbb{R}^3$ *having an invariant hyperbolic, parabolic or elliptic cylinder*?, and *for such polynomial differential systems what are their phase portraits on such invariant cylinders*?

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

A polynomial differential system in $\mathbb{R}^3$ is a differential system of the form

$$\begin{aligned} \dot{x} &= P(x, y, z), \\ \dot{y} &= Q(x, y, z), \\ \dot{z} &= R(x, y, z), \end{aligned} \tag{1}$$

where P, Q and R are real polynomials in the variables x, y and z, and the dot denotes derivative with respect to the time t. The degree of the polynomial differential system (1) is the maximum of the degrees of the polynomials P, Q and R.

Three of the nicer surfaces in $\mathbb{R}^3$ are the hyperbolic, parabolic and elliptic cylinders. We say that one of these cylinders is *invariant* under the flow of the differential system (1) if for every orbit $(x(t), y(t), z(t))$ of the differential system (1) having a point on that cylinder the whole orbit is contained in it.

Two natural questions about the invariant cylinders of a polynomial differential system (1) are: *What are the easiest polynomial differential systems* (1) *in* $\mathbb{R}^3$ *having an invariant cylinder*, and *for such polynomial differential systems what are their phase portraits on the invariant cylinder*? The objective of this paper is to give an answer to these two questions.

Let U be an open and dense set in $\mathbb{R}^3$. We recall that a C^1 function $H : U \to \mathbb{R}$ which is non-locally constant is a *first integral* of the differential system (1) if H is constant on all the solutions $(x(t), y(t), z(t))$ contained in U . In other words, on the solution $(x(t), y(t), z(t))$ we have that

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}P + \frac{\partial H}{\partial y}Q + \frac{\partial H}{\partial z}R = 0. \tag{2}$$

2010 *Mathematics Subject Classification.* Primary 34C05, 34A34.
Key words and phrases. Polynomial differential systems in $\mathbb{R}^3$, hyperbolic cylinder, parabolic cylinder, elliptic cylinder.