# STABILITY OF EQUILIBRIUM POINTS IN THE SPATIAL RESTRICTED $N+1$-BODY PROBLEM WITH MANEV POTENTIAL

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Abstract. We study the dynamics of an infinitesimal mass under the gravitational attraction of $N-1$ primaries arranged in a planar ring configuration plus the influence of the central mass with a Manev potential $(-1/r + e/r^2)$, $e \neq 0$, where $e$ is a parameter related to the oblaticity or radiation source (according to the sign of the parameter $e$). Specifically, we investigate the relative equilibria of the infinitesimal mass and their linear stability as functions the parameter $e$ and the mass parameter $\beta$, the ratio of mass of the central body to the mass of one of $N-1$ remaining bodies. We also prove the nonexistence of binary collisions between the central body and the infinitesimal mass.

**Key words:** Restricted N+1-body problem, Manev potential, Equilibrium points, Stability

**MSC codes:** 70F10, 70F15, 37C25

## 1. Introduction

The two body problem with a quasi–homogeneous potential of the form $-(a/r - e/r^2)$, where $r$ is the distance between the two bodies, and $a$, $e$ are real constants, was considered by Newton in his work Philosophiae Naturalis Principia Mathematica (Book I, Article IX, Proposition XLIV, Theorem XIV, Corollary 2). One of the reasons to add the term $e/r^2$ to the gravitational attraction $(-a/r)$ was the impossibility to explain the Moon's apsidal motion within the framework of the inverse-square force law, although the model was abandoned in favor of the classical Newtonian potential. Manev in 1924, [15], proposed a similar corrective term in order to maintain classical mechanics and offering at the same time good explanations of the observed phenomena as in the relativity theory. For instance, when $a$ is positive and $e$ is negative, the corrective term is good enough to explain the perihelion advance of Mercury.

In this work we consider the motion in a three-dimensional space of an infinitesimal mass $P$ under the gravitational attraction of $N = n+1$ point masses, $P_0, P_i$, $i = 1, \ldots, n$ called *primaries*. We assume that the potential generated by the primary $P_0$ is a Manev potential $(-1/r + e/r^2)$, with parameter $e$, and that the gravitational attraction due to $P_i$, $i = 1, \ldots, n$ is Newtonian $-1/r$. We also shall assume that the $n$-primaries $P_i$ $(i = 1, \ldots, n)$ are in a $n$-gon configuration, that is, the bodies $P_i$, $i = 1, \ldots, n$ have the same mass $m_i = m$, for all $i = 1, \ldots, n$, and are located symmetrically with respect to the central body $P_0$, of mass $m_0 = \beta m$, which is at the center of mass of the system. $P_0$ will also be called the central body, and $P_i$, $i = 1, \ldots, n$ the peripherals, as in the Maxwell ring model. In an inertial reference system the peripheral bodies move in a circular

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*Date*: April 20, 2023.

Second author is supported by the Spanish grant PGC2018-100928-B-I00.
Third author is supported by MINECO grants MTM2013-40998-P,MTM2016-77278-P FEDER and AGAUR grant 2014 SGR 568.

orbit around $P_0$ with angular velocity $\omega$. This problem will be called Maxwell's ring restricted $(N+1)$-body problem with Manev potential or shortly, Manev R$(N+1)$BP.

The case $e = 0$, shortly, the classical Maxwell model was considered by Scherees in [19] several aspects of the dynamcis were studied, such as , Hill stability, invariant transformations, equilibrium points and their stability, and periodic orbits. After that, Kalvouridis in [14] for the planar case formulate the general equations of motion and studied the stationary solutions and the zero-velocity contours for various values of $n$.

We emphasize that the parameter $e \in \mathbb{R}$ models several problems, for example, when the central body of the ring is no longer spherical, but an ellipsoid of revolution (spheroid). According to [11], [12] the parameter $e$ is associated with flattening, in natural bodies like planets, the spheroid is flattened $e < 0$, but also we can think of artificial bodies and assume they are prolates, in that case $e > 0$. In general, this fact is seen more used in potentials of the Schwarzschild type ($A/r - e/r^3$, introduced in 1998 by Mioc and Savinski in [17]). We consider that the central body is a source of radiation, repulsive if $e > 0$ and attractive if $e < 0$, and then the effect of radiation can be modeled in a similar way to the flattened ellipsoid (see, for example, [13]).

In Fakis and Kalvouridis [11] (2013) the authors study numerically some aspects of the dynamics of a small body under the action of Maxwell-type $N$-body system with a spheroidal central body. As for example, the equilibrium locations and their parametric dependence, as well as the zero-velocity curves and surfaces for the planar motion, and the evolution of the Hill's regions. The non-sphericity of the central body is described by a Manev potential, as presented in this work. See also Elipe et al. [12] (2007), Arribas et al. [4] (2003) and Arribas et al. [5] (2007). In Alavi and Razmi [1] (2015), such a correction term in a Newtonian potential, with $e > 0$ (that represents a repulsive centripetal force), is used in disk galaxies evolution. Also, in Mioc and Stoica [16] (1997) the Manev-type potential is considered in the frame of a two-body problem. The spatial restricted four body problem (case $n = 2$) with repulsive Manev potential ($e > 0$) was studied from an analytical point of view in [10]. For the planar case and $n = 7$, a particular numerical study on the number of equilibria and the bifurcations that depend on the Manev parameter is made in [3]. We found that in [12] was studied the existence of some symmetric periodic solutions in the planar case using numerical methods. For the spatial case with general $n$, an analytical study of the existence of periodic solution families around the central body and far from the primaries was studied by Ascencio and Vidal. In [6] the authors proved the existence of symmetric periodic solutions. Then, in [7] they proved the existence of periodic solutions (not necessarily symmetric), where they also guaranteed the existence of KAM tori that enclosed them.

The main purpose of this paper is to study important aspects of the dynamics of the spatial restricted $(N+1)$-body problem with repulsive or attractive Manev potential from an analytical point of view, for any quantity of peripherals $n$. Initially, we characterize the symmetries of the associated Hamiltonian function. On the other hand, for the repulsive case, that is, $e > 0$ we prove that, due to the repulsive force emanating from the central body, it is not possible to have a binary collision between the infinitesimal mass and the central body in the Manev R$(N+1)$BP. We prove that any equilibrium point must lie on the lines of symmetries of the regular polygon formed by the peripheral bodies, or on the $z$-axis. Using this information we are able to determine the type of equilibrium points and the number of them as functions of the parameters $\beta$ and $e$. Bifurcation parameters are characterized. After that, several general results concerning the type of stability of each equilibria are proved analytically.

The paper is organized as follows: in Section 2 we point out the equation of motions, the admissible values of the Manev parameter $e$, and the symmetries. We also prove the nonexistence of binary collisions between the central body and one of the infinitesimal mass. Section 3 is devoted to the observe that any planar equilibrium point must lie on the symmetries lines of the regular polygon formed by the peripherals. Using this information we are able to determine the type of equilibrium points and the number of them as function of the parameters $\beta$ and $e$. Bifurcation parameters are characterized. In Section 4 the linear stability of each equilibrium point is given. Finally, in Section 6 we introduce some technical lemmas that are necessary for the proof of our results.

## 2. Statement of the problem and main features

In this section we derive the equations of motion of the Manev R($N+1$)BP as follows. Consider $N$+1 bodies, $P_i$, with positive masses $m_i$, in an inertial frame moving under their mutual Newtonian gravitational attraction, plus a Manev perturbation coming from body $P_0$. The potential generated by the $N+1$ bodies is given by

$$U = \sum_{0\le i<j\le N} \frac{\mathcal{G}m_i m_j}{||q_i - q_j||} - \sum_{j=1}^{N} \frac{\mathcal{G}m_0 m_j B}{||q_0 - q_j||^2}, \tag{1}$$

where $q_i$ is the position of $P_i$, $i = 0, 1, \ldots, N$, $\mathcal{G}$ is the Gaussian constant of gravitation and $B$ is the corrective coefficient corresponding to Manev potential.

If we consider that the particle $P = P_N$ with position $q = q_N$ is small, $m_N \approx 0$, so that its influence on the other bodies can be neglected, the equations of motion of a restricted $N+1$-body problem are

$$\begin{aligned} \ddot{q}_0 &= \sum_{j=1}^{n} \left( \frac{\mathcal{G}m_j(q_j - q_0)}{||q_0 - q_j||^3} - \frac{2\mathcal{G}m_j B(q_j - q_0)}{||q_0 - q_j||^4} \right), \\ \ddot{q}_i &= \sum_{j=0,\ j\neq i}^{n} \frac{\mathcal{G}m_j(q_j - q_i)}{||q_i - q_j||^3} - \frac{2\mathcal{G}m_0 B(q_0 - q_i)}{||q_0 - q_i||^4},\ i = 1, \ldots, n, \\ \ddot{q} &= \sum_{j=0}^{n} \frac{\mathcal{G}m_j(q_j - q)}{||q - q_j||^3} - \frac{2\mathcal{G}m_0 B(q_0 - q)}{||q - q_0||^4}. \end{aligned} \tag{2}$$

where $N = n + 1$. The first $n + 1$ equations correspond to the motion of the primaries and are uncoupled, in the sense that they can be solved independently from the last one. The last one corresponds to the motion of the infinitesimal particle, and in order to solve it a solution of the first $n + 1$ equations is required.

We impose the following solution for the primaries. We place $P_0$, called central primary, at the origin and the remaining bodies, called peripherals, $P_i$, $i = 1, \ldots, n$, with equal masses $m_i = m$, $i = 1, \ldots, n$, at the vertices of a regular polygon with center at $P_0$, and moving around it, in a plane, with constant angular velocity $\omega$. Then

$$q_j(t) = de^{iwt} e^{i\frac{2\pi(j-1)}{n}},\ j = 1, \ldots, n, \tag{3}$$

where $d$ is the radius of the polygon. Substituting into the first $n + 1$ equations in (2), and introducing the mass parameter $\beta = m_0/m$, we obtain the following two algebraic equations

$$0 = de^{i\omega t}\mathcal{G}m \left( \sum_{j=1}^{n} \frac{e^{i\frac{2\pi(j-1)}{n}}}{d^3} - \frac{2Be^{i\frac{2\pi(j-1)}{n}}}{d^4} \right), \tag{4}$$

$$-\omega^2 de^{i\omega t} = \mathcal{G}mde^{i\omega t} \left( \sum_{j=2}^{n} \frac{e^{i\frac{2\pi(j-1)}{n}} - 1}{d_j^3} - \frac{\beta}{d^3} + \frac{2B\beta}{d^4} \right), \tag{5}$$

where $d_j = ||q_1 - q_j||$ is the distance between the peripherals $P_1$ and $P_j$, with $j = 2, \dots, n$.

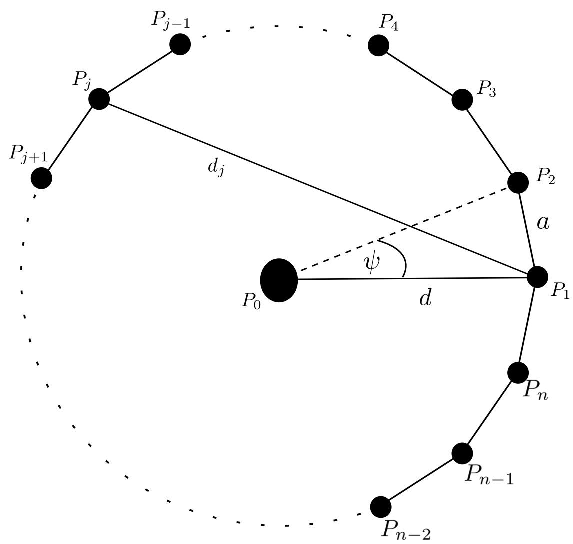



FIGURE 1. The "ring" configuration of the $n+1$ primaries, where $\psi = 2\pi/n$, $d$ is the radius of the ring and $a$ the side of the regular polygon, related by (6).

On one hand, using trigonometric identities it is not difficult to see that $\sum_{j=1}^{n} e^{i\frac{2\pi(j-1)}{n}} = 0$, so equation (4) is satisfied trivially. On the other hand, using the geometry of the configuration (see Figure 1) we have that

$$d = \frac{a}{2\sin(\pi/n)} = \frac{a}{\rho}, \qquad d_j = \frac{2a}{\rho}\sin\left((j-1)\frac{\pi}{n}\right), \tag{6}$$

where $a$ is the side of the regular polygon and $\rho = 2\sin(\pi/n)$. Substituting (6) into (5) and defining $e = B/a$, *the Manev parameter*, we have that

$$w^2 = -\mathcal{G}m\sum_{j=2}^{n}\frac{e^{i\frac{2\pi(j-1)}{n}} - 1}{d_j^3} + \mathcal{G}m\frac{\beta\rho^3}{a^3} - \mathcal{G}m\frac{2\beta e\rho^4}{a^3}. \tag{7}$$

Clearly, using the symmetry of the configuration and (6)

$$\begin{aligned} \Im\left(\sum_{j=2}^{n}\frac{e^{i\frac{2\pi(j-1)}{n}} - 1}{d_j^3}\right) &= 0, \\ \Re\left(\sum_{j=2}^{n}\frac{e^{i\frac{2\pi(j-1)}{n}} - 1}{d_j^3}\right) &= \frac{-\rho}{a^3}\sum_{j=2}^{n}\frac{\sin^2(\pi/n)}{\sin((j-1)\pi/n)}. \end{aligned}$$

We define

$$\Lambda = \sum_{i=2}^{n}\frac{\sin^2(\pi/n)}{\sin[(i-1)(\pi/n)]}, \tag{8}$$

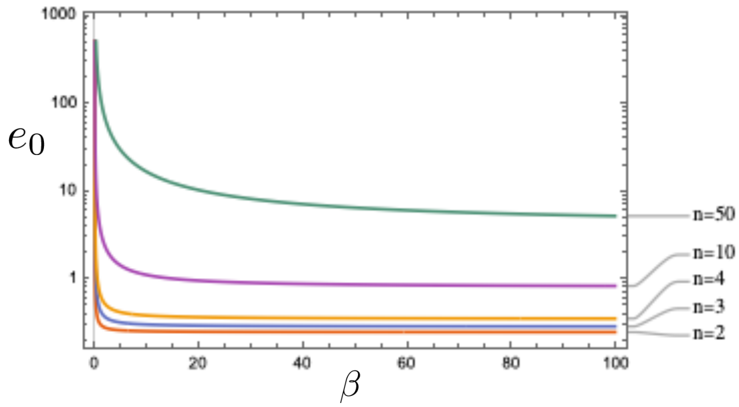



FIGURE 2. Evolution of $e_0$, defined in (11), as a function of the mass ratio $\beta$ for different values of $n$ (log scale).

so that equation (7) writes

$$w^2 = \mathcal{G}m\left(\frac{\rho\Lambda}{a^3} + \frac{\beta\rho^3}{a^3} - \frac{2\beta e\rho^4}{a^3}\right),$$

or equivalently

$$\frac{\mathcal{G}m}{a^3\omega^2} = \frac{1}{\Delta}, \tag{9}$$

where

$$\Delta = \rho(\Lambda + \beta\rho^2 - 2\beta e\rho^3). \tag{10}$$

Therefore the configurations were $n$ bodies are at the vertices of a regular polygon, rotating with constant angular velocity, plus a mass with a Manev potential at the center is solution of the $n+1$-body problem provided that equation (9) is satisfied. Equation (9) can be interpreted as generalized third Kepler law.

Notice that, from (9), $\Delta$ must always be positive, which gives an upper bound on the Manev parameter $e$.

**Definition 2.1.** *For each fixed integer $n \geq 2$ and mass ratio $\beta > 0$, the admissible values of the Manev parameter $e$ are the values such that*

$$e < e_0 := \frac{\Lambda + \beta\rho^2}{2\beta\rho^3}, \tag{11}$$

*where $\Lambda$ is given in (8).*

In Figure 2 we see the evolution of $e_0$ as a function of $\beta$ for different values of $n$. Clearly, the greater the number of peripherals, the greater the curve $e_0(\beta)$. Thus, if the Manev parameter is big, either the mass ratio $\beta$ is small or the number of peripherals is big enough.

Introducing the Manev parameter $e$ and the mass ratio $\beta$ in the last equation of system (2), the motion of the infinitesimal particle $P$ is given by

$$\ddot{q} = \mathcal{G}m\left(-\frac{\beta}{r_0^3}q + \frac{2e\beta\, a}{r_0^4}q + \sum_{i=1}^{n}\frac{q_i - q}{r_i^3}\right), \tag{12}$$

where $q_i(t)$ are given in (3), $r_i(t)$ is the distance between $P$ and the $i$th primary, and parameters $\mathcal{G}$, $m$, $\beta$, $e$, $a$ must satisfy equation (9). By scaling distances by $q_i^* = \frac{q_i}{a}$, $i = 0, \ldots, n$, $q^* = \frac{q}{a}$ and time by $t^* = \omega t$, and using the identity (9) we obtain the equations of the restricted Manev problem in the inertial frame (for simplicity we drop the $^*$ notation)

$$\ddot{q} = \frac{1}{\Delta}\left( -\frac{\beta q}{r_0^3} + \frac{2e\beta q}{r_0^4} + \sum_{i=1}^{n} \frac{q_i - q}{r_i^3} \right). \tag{13}$$

Notice that, with the reescaling, the peripherals are located in an $n$-gon of side 1 with radius $1/\rho$ and rotating periodically with period $2\pi$.

We change to a rotating system $Oxyz$, that rotates with angular velocity equal to 1, so that the peripherals are contained in the plane $z = 0$ at fixed positions $(x_i, y_i, 0)$, where

$$x_j = \frac{1}{\rho}\cos\left(\frac{2\pi(j-1)}{n}\right), \quad y_j = \frac{1}{\rho}\sin\left(\frac{2\pi(j-1)}{n}\right), \quad j = 1, \ldots, n. \tag{14}$$

Then, the motion of the infinitesimal particle in the rotating system (see Figure 3) is described by the following system of second-order differential equations:

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x, \\ \ddot{y} + 2\dot{x} &= \Omega_y, \\ \ddot{z} &= \Omega_z, \end{aligned} \tag{15}$$

where $\Omega_\xi = \frac{\partial \Omega}{\partial \xi}$,

$$\Omega(x, y, z) = \frac{1}{2}(x^2 + y^2) + V(x, y, z) \tag{16}$$

$$V(x, y, z) = \frac{1}{\Delta}\left[ \beta\left( \frac{1}{r_0} - \frac{e}{r_0^2} \right) + \sum_{j=1}^{n} \frac{1}{r_j} \right], \tag{17}$$

and

$$r_0 = (x^2 + y^2 + z^2)^{1/2}, \quad r_j = [(x - x_j,)^2 + (y - y_j)^2 + z^2]^{1/2}, \quad j = 1, \ldots, n. \tag{18}$$

These equations are the same as in [11].

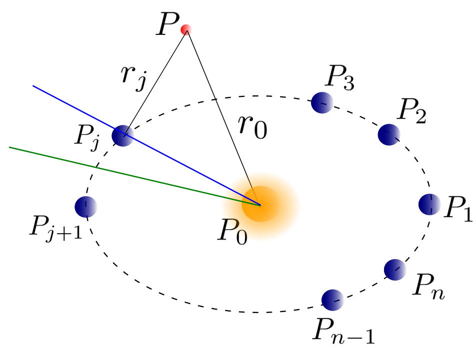



FIGURE 3. The configuration of the problem. $P$ is the small body and $P_i$, $i = 0, 1, 2, \ldots, n$ are the primaries.

The phase space associated to system (15) (as a first order differential system) is

$$\mathcal{M} = \Big\{(x, y, z, \dot{x}, \dot{y}, \dot{z}) \in \big(\mathbb{R}^3 \setminus \{(0,0,0), (x_i, y_i, 0) : i = 1, \dots, n)\}\big) \times \mathbb{R}^3\Big\}.$$

We remark that the problem has two invariant subspaces: the subspace $z = \dot{z} = 0$, named Planar Manev R(N+1)BP, and the $z$-axis, named Rectilinear Manev R(N+1)BP.

Next, we highlight some main properties of the model.

### 2.1. Rotation and symmetries.

The system (15) admits the following rotation

$$R: (x, y, z, \dot{x}, \dot{y}, \dot{z}) \to \begin{array}{l}(\cos(\psi)x - \sin(\psi)y, \sin(\psi)x - \cos(\psi)y, z, \\ \cos(\psi)\dot{x} - \sin(\psi)\dot{y}, \sin(\psi)\dot{x} + \cos(\psi)\dot{y}, \dot{z}),\end{array} \tag{19}$$

with $\psi = 2\pi/n$. In the next section we will see that the use of previous rotation will simplify the study of the equilibrium points.

In addition, the system (15) admits the following time reversal symmetries:

$$\begin{array}{ll} S_1: & (x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \to (x, -y, -z, -\dot{x}, \dot{y}, \dot{z}, -t), \\ S_2: & (x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \to (x, -y, z, -\dot{x}, \dot{y}, -\dot{z}, -t), \end{array} \tag{20}$$

for all $n$, and

$$\begin{array}{ll} S_3: & (x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \to (-x, y, -z, \dot{x}, -\dot{y}, \dot{z}, -t), \\ S_4: & (x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \to (-x, y, z, \dot{x}, -\dot{y}, -\dot{z}, -t), \end{array} \tag{21}$$

for $n$ even. They have been used to prove the existence of comet and Hill periodic orbits around the primaries (see [6]).

### 2.2. Jacobi constant.

Similarly to the classical circular restricted three-body problem, the system (15) possesses the first integral, known as Jacobi constant, given by

$$C = 2\Omega(x, y, z) - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2). \tag{22}$$

Using the above first integral, we can prove that in the repulsive case, it is not possible to have a binary collision between the infinitesimal mass and the central body in the Manev R($N+1$)BP. This is consequence of the following result.

**Theorem 2.1.** *For each integer $n \geq 2$, $\beta > 0$ and an admissible $e > 0$, a solution of the restricted Manev R(N+1)BP (15) must satisfy*

$$\liminf_{t \to \pm\infty} r_0(t) > 0,$$

*where $r_0^2 = x^2 + y^2 + z^2$.*

*Proof.* Consider $\gamma(t)$ a solution of (15). Then by (22), there exists a constant $C \in \mathbb{R}$ such that $C(\gamma(t)) = C\ \forall\ t$. Suppose that $\liminf_{t \to +\infty} r_0(t) = 0$ (analogously when $t \to -\infty$). Then, using (17), there exists a sequence $t_n \longrightarrow +\infty$ such that

$$\lim_{n \to \infty} C(\gamma(t_n)) = -\infty,$$

which is a contradiction. $\square$

## 3. Equilibrium points

The equilibrium points of the Manev R(N+1)BP (15) correspond to the points $(x, y, z, 0, 0, 0) \in \mathcal{M}$ such that

$$\begin{aligned} x - \frac{1}{\Delta}\left[\beta\left(\frac{1}{r_0^3} - \frac{2e}{r_0^4}\right)x + \sum_{i=1}^{n}\frac{x - x_i}{r_i^3}\right] &= 0, \\ y - \frac{1}{\Delta}\left[\beta\left(\frac{1}{r_0^3} - \frac{2e}{r_0^4}\right)y + \sum_{i=1}^{n}\frac{y - y_i}{r_i^3}\right] &= 0, \\ z\left[\beta\left(\frac{1}{r_0^3} - \frac{2e}{r_0^4}\right) + \sum_{i=1}^{n}\frac{1}{r_i^3}\right] &= 0. \end{aligned} \tag{23}$$

Since any equilibrium point is determined by the position $(x, y, z)$ of the infinitesimal mass, from now we represent them only by the position vector.

In the following result we characterize the location of the equilibrium points.

**Theorem 3.1.** *Consider the Manev R(N+1)BP (15) for a fixed $n \geq 2$, $\beta > 0$ and an admissible $e$. The equilibrium points in the $z = 0$ plane lie on the lines $y = \tan(\frac{i\pi}{n})\,x$, $i = 1, \ldots, n$. In the spatial case $z \neq 0$, for $e > 0$ the equilibrium points are located on the $z$-axis, while for $e \leq 0$ there are no equilibrium points.*

*Proof.* For $n = 2$, the result is already proved in [10]. Then, we consider $n \geq 3$.

We first consider the planar case $z = 0$. The equations of the equilibrium points given in (23) are reduced to

$$\begin{aligned} x + V_x &= 0, \\ y + V_y &= 0, \end{aligned} \tag{24}$$

where $V$ is given in (17). The system (24) implies

$$yV_x - xV_y = 0, \quad \longrightarrow \quad \sum_{i=1}^{n}\frac{xy_i - yx_i}{r_i^3} = 0, \tag{25}$$

where $x_i$ and $y_i$ are given in (14). Denote by $\varphi_i = \frac{2\pi(i-1)}{n}$, $i = 1, \ldots, n$. Introducing polar coordinates $x = -r\cos\theta$, $y = r\sin\theta$, for a fixed $r > 0$ the equation (25) can be written as $F(\theta) = 0$ where

$$F(\theta) := \sum_{i=1}^{n}\frac{\sin(\theta + \varphi_i)}{r_i^3} = \frac{\sin(\theta)}{r_1^3} + \sum_{i=1}^{n-1}\frac{\sin(\theta + \varphi_{i+1})}{r_{i+1}^3},$$

where $r_i^2 = r^2 + 2r\cos(\theta + \varphi_i) + 1$. Notice that if we consider $r_i$ as a function of $\theta$, then $r_i(\theta) = r_1(\theta + \varphi_i)$. Therefore, $F(\theta)$ can be written as

$$F(\theta) = f(\theta) + \sum_{i=1}^{n-1} f(\theta + iT),$$

with $f(\theta) = \frac{\sin(\theta)}{r_1^3(\theta)}$ and $T = \frac{2\pi}{n}$. In this way, $F(\theta)$ satisfies the hypothesis of Lemma 6.1 (see Appendix 6), and $F(\theta) = 0$ if and only if $\theta = \frac{k\pi}{n}$, $k \in \mathbb{Z}$, which completes the proof in the case $z = 0$.

Next we consider $z \neq 0$. In this case the system (23) can be rewritten as

$$(26)\qquad \begin{aligned} \Delta\, Q\, x &= -\sum_{i=1}^{n} \frac{x_i}{r_i^3}, \\ \Delta\, Q\, y &= -\sum_{i=1}^{n} \frac{y_i}{r_i^3}, \\ (1-Q)\, z &= 0, \end{aligned}$$

with $Q = 1 - \frac{1}{\Delta}\left(\beta\left(\frac{1}{r_0^3} - \frac{2e}{r_0^4}\right) + \sum_{i=1}^{n} \frac{1}{r_i^3}\right)$. Since $z \neq 0$, then $Q = 1$ and we have that

$$\beta\left(\frac{1}{r_0^3} - \frac{2e}{r_0^4}\right) + \sum_{i=1}^{n} \frac{1}{r_i^3} = 0.$$

Clearly, this equation does not have solution if $e < 0$, so there are not equilibrium points on the $z$-axis when $e < 0$.

When $e > 0$, system (26) is reduced to

$$(27)\qquad \begin{aligned} \Delta\, x &= -\sum_{i=1}^{n} \frac{x_i}{r_i^3}, \\ \Delta\, y &= -\sum_{i=1}^{n} \frac{y_i}{r_i^3}. \end{aligned}$$

Using first Lemma 6.2 we have that if $y \neq 0$ both sides of the second equation in (27) have different sign, so we have a contradiction and $y = 0$. Then, introducing $y = 0$ in the first equation, and using Lemma 6.3 we have that if $x \neq 0$ both sides of the equation have different sign, so again we have a contradiction. Therefore, $x = 0$ and the equilibrium points with $z \neq 0$ must be located on the $z$-axis. This completes the proof. □

Notice that the lines $y = \tan(\frac{i\pi}{n})\, x$, $i = 1, \ldots, n$, are lines of symmetry of the configuration of the primaries, and using the rotational symmetry (19) it is enough to study the localization and number of equilibrium points on the half lines $\mathcal{R}$ and $\mathcal{L}$ defined bellow (see Figure 4). Analogously, by symmetry with respect the plane $z = 0$, it is also enough to study the spatial equilibria for $z > 0$.

**Definition 3.1.** *We denote by $\mathcal{R}$ and $\mathcal{L}$ the half lines on the $z = 0$ plane:*

$$\begin{aligned} \mathcal{R} &= \{y = z = 0, x > 0\}, \\ \mathcal{L} &= \{z = 0, y = \tan(\pi/n)x, x > 0\}. \end{aligned}$$

*We also denote $\mathcal{R}_1 = \{y = z = 0, x > 1/\rho\}$ and $\mathcal{R}_2 = \{y = z = 0, 0 < x < 1/\rho\}$.*

Notice that $\mathcal{R}$ contains one peripheral at $(1/\rho, 0)$, whereas $\mathcal{L}$ is the bisector between the lines containing $P_1$ and $P_2$. We study separately the number and location of equilibrium points on $\mathcal{R}$ and $\mathcal{L}$, see Figure 4.

**Definition 3.2.** *The equilibrium points that lie on $\mathcal{R}$ and $\mathcal{L}$ are denoted by $L_p$ and $L_m$ respectively, and the equilibrium points on the $z$-axis, with $z > 0$ by $L_z$.*

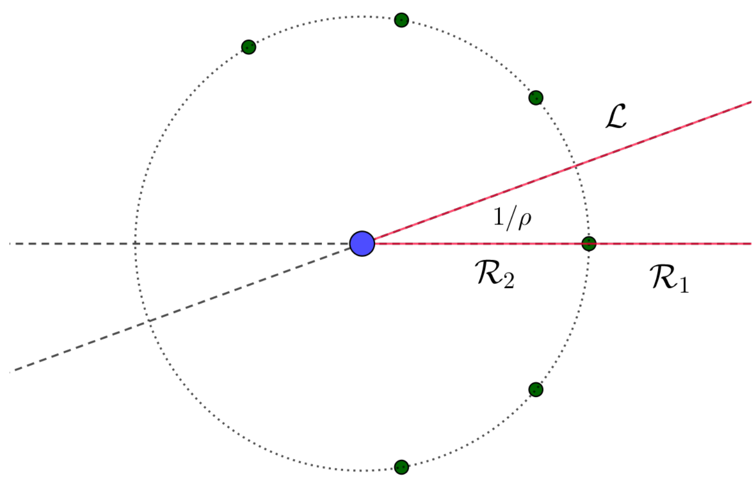



FIGURE 4. It is enough to study the equilibrium points on $\mathcal{R}$ and $\mathcal{L}$ (see Definition 3.1). All the other equilibrium points are obtained applying rotations of angle $2\pi/n$. The dotted circles indicates the location of the peripherals.

3.1. **Equilibrium points on the $z$-axis with $e > 0$.** From (23) an equilibrium point on the positive $z$-axis is a solution $z > 0$ of

$$\beta\left(\frac{1}{z^3} - \frac{2e}{z^4}\right) + \frac{n}{(1/\rho^2 + z^2)^{3/2}} = 0. \tag{28}$$

The following result shows the existence of only one equilibrium point on the $z$-axis with $z > 0$ and gives a bound on its location.

**Theorem 3.2.** *Consider the Manev R(N+1)BP (15) for a fixed $n \geq 2$, $\beta > 0$ and an admissible $e > 0$. Then there exists a unique equilibrium point on the positive $z$-axis, $L_z = (0, 0, \bar{z})$. Furthermore, $0 < \bar{z} < 2e$.*

*Proof.* Consider the auxiliary functions

$$h_1(z) = \beta\left(\frac{1}{z^3} - \frac{2e}{z^4}\right) \quad \text{and} \quad h_2(z) = -\frac{n}{(1/\rho^2 + z^2)^{3/2}}.$$

From (28) an equilibrium point on the positive $z$ axis is a solution of the equation $h_1(z) = h_2(z)$. On one hand, we have that $\lim_{z\to 0^+} h_1(z) = -\infty$, $h_1(z) < 0$ and $h_1'(z) > 0$ for $0 < z < 2e$, and $h_1(z) > 0$ for $z > 2e$. On the other hand, $h_2(z) < 0$ and $h_2'(z) > 0$ for $z > 0$ (see Figure 5). Then, it is straightforward that there exists a unique positive solution of (28) located in $(0, 2e)$. □

**Proposition 3.1.** *Let $L_z = (0, 0, \bar{z})$, $\bar{z} = \bar{z}(e, \beta)$, be the equilibrium point given in Theorem 3.2. Then,*

$$\lim_{e\to 0^+} \bar{z} = 0, \quad \lim_{\beta\to 0^+} \bar{z} = 0, \quad \textit{and} \quad \lim_{\beta\to +\infty} \bar{z} = 2e.$$

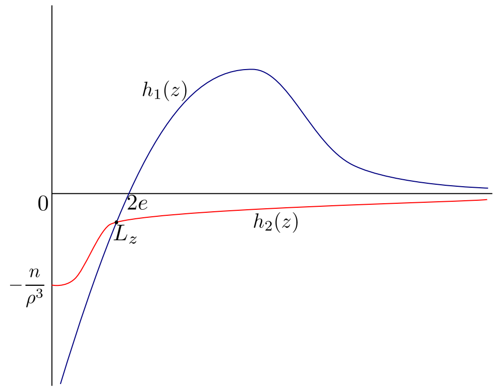



FIGURE 5. Graphics associated to the functions $h_1$ and $h_2$ (see Theorem 3.2). The intersection of the curves show the existence of the equilibrium on the $z$-axis.

*Moreover $\bar{z}$ is an increasing function of $\beta$ and $e$, and $\bar{z} = 2e + \mathcal{O}(e^3)$ for all $\beta$.*

*Proof.* The first limit is obtained from the upper and lower bounds of $\bar{z}$. To obtain the second limit, notice that using (28), we can write (for any fixed value of $e$) $\beta$ as a function of $\bar{z}$ as

$$\beta = \frac{n\bar{z}^4}{(2e - \bar{z})(1/\rho^2 + \bar{z}^2)^{3/2}}.$$

Using Taylor expansion we get $\beta = \frac{\rho^3 n}{2e}\bar{z}^4 + O(\bar{z}^5)$. The third limit is obtained directly dividing equation (28) by $\beta$.

For the monotonicity, notice that the function $h_1$ in the proof of Theorem 3.2 is decreasing in the variables $\beta$ and $e$, which implies that $\bar{z}$ is increasing in $\beta$ and $e$.

Finally, to prove that

$$\lim_{e \to 0^+} \frac{\partial \bar{z}}{\partial e} = 2$$

we introduce in equation (28) the variable $u^2 = 1 + \rho^2 z^2$ and the rational parametrization

$$z = \frac{s^2 - 1}{2s}, \quad u = \frac{s^2 + 1}{2s}, \quad s > 1$$

to obtain

$$\beta(s^2 + 1)^3(s^2 - 4e\rho s - 1) + n(s^2 - 1)^4 = 0.$$

It is not difficult to see that the above equation has a unique solution for $s > 1$, $\bar{s} = \bar{s}(e, \beta)$, which satisfies $\bar{s} < 2e\rho + \sqrt{1 + 4e^2\rho^2}$. Deriving with respect $e$ we have

$$\lim_{e \to 0^+} \frac{\partial \bar{s}}{\partial e} = 2\rho,$$

from which the claim follows. □

Notice that $2e$ is a sharp bound when $\beta$ is bigger or $e$ is small. In Figure 6 we show the evolution of the location of $L_z$ as a function of $e$ for different values of $\beta$ and $n$. As we have proved in the previous proposition the curves are tangent to the line $\bar{z} = 2e$.

**Remark 3.1.** *In [10] was proved that $\min\{e, \beta e\} < \bar{z}$ when $n = 2$. Straightforward argument shows that it is also true for $n = 3, 4$. As we can see at Figure 6 the lower bound fails for bigger values of $n$.*

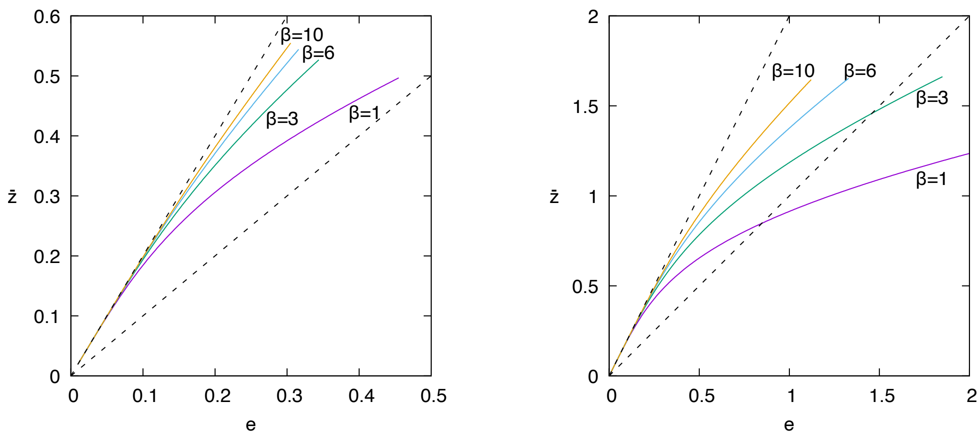



FIGURE 6. Curves $(e, \bar{z}(e))$ for $e \in (0, e_0)$ (see Theorem 3.2) for different fixed values of $\beta = 1, 2, \ldots, 10$ and $n = 3$ (left) and $n = 10$ (right). The dashed lines correspond to the lines $\bar{z} = e$ and $\bar{z} = 2e$ (see Proposition 3.1).

3.2. **Equilibrium points on the half line $\mathcal{R}$.** From (23), an equilibrium point on the positive $x$-axis must be a solution of

$$\Delta x^3 + \frac{2\beta e}{x} - \beta = x^2 \sum_{i=1}^{n} \frac{x - x_i}{((x - x_i)^2 + y_i^2)^{3/2}}. \tag{29}$$

In order to solve the above equation we use the auxiliary functions

$$f_1(x) = \Delta x^3 + \frac{2\beta e}{x} - \beta, \tag{30}$$

and

$$f_2(x) = x^2 \sum_{i=1}^{n} \frac{x - x_i}{((x - x_i)^2 + y_i^2)^{3/2}}, \tag{31}$$

defined for $x > 0$. It is clear that solving equation (29) is equivalent to solve $f_1(x) = f_2(x)$ for $x > 0$.

**Definition 3.3.** *For $0 < e < e_0$, let $x^* = x^*(e) = \left(\frac{2\beta e}{3\Delta}\right)^{1/4}$ be the minimum of $f_1$ given in (30).*

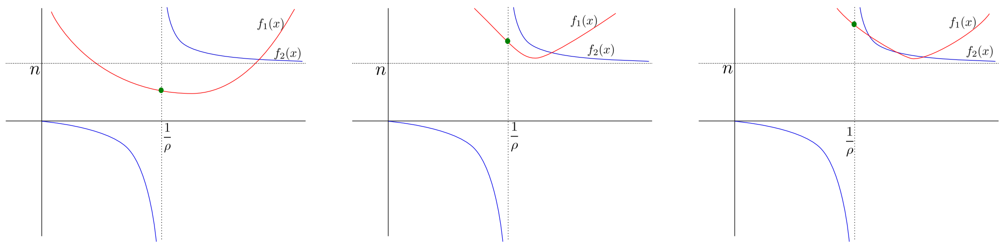
Next result states the number of equilibrium points along $\mathcal{R}_1$, that is, when $x > 1/\rho$ at the right hand side of the peripheral.

**Theorem 3.3.** *For any fixed value of $n$ and $\beta > 0$:*

(1) *If $0 < e < e_0$ there exists at least one equilibrium point on $\mathcal{R}_1$ denoted by $L_{p_1} = (\bar{x}_1, 0, 0)$. In addition, if $n \le 472$ this equilibrium is unique. Moreover, $\bar{x}_1 \ge \max\{1/\rho, x^*\}$, where $x^*$ is given in Definition 3.3.*

(2) *If $e \le 0$, there exists exactly one equilibrium point on $\mathcal{R}_1$, $L_{p_1}$.*

*Proof.* An equilibrium point on $\mathcal{R}_1$ satisfies the equation $f_1(x) = f_2(x)$. The existence of at least one equilibrium point for any admissible $0 < e < e_0$ follow observing that the curve $f_1(x)$ and $f_2(x)$ intersect in at least one point for $x > 1/\rho$ (see Figure ). This affirmative is consequence of Lemmas




(a) $f_1(1/\rho) < n$ (b) $f_1(1/\rho) > n$ (c) $f_1(1/\rho) > n$

Figure 7. Examples of graphics associated to the function $f_1$ and $f_2$ (see Theorem 3.3). The intersection of the curves show the existecne of equilibrium points on the $x$-axis.

6.4 and 6.5. Using that $f_1(x^*) < f_1(1/\rho) < f_2(x)$ for any $x > 1/\rho$, we obtain the lower bound for $\bar{x}_1$. The uniqueness follows observing that by Lemma 6.4 item (2).(iv) the function $f_1(1/\rho)$ is an increasing function of $n$. Thus, by simple inspection we arrive that when $n \leq 472$ then $f_1(1/\rho) < n$ (note that for $n = 472$ the respective value $f_1(1/\rho) \approx 471.956882$), this guarantee the uniqueness of the intersection point between the curves $f_1(x)$ and $f_2(x)$ for $x > 1/\rho$ (see Figure 7). □

**Proposition 3.2.** *For any $\beta > 0$ and admissible $e$, let $L_{p_1}$ be the equilibrium point given in Theorem 3.3. Then:*

1. $\lim_{e\to 0} \bar{x}_1$ *is finite;*
2. $\lim_{\beta\to 0} \bar{x}_1 = \bar{x}_{1_0}$*, where $\bar{x}_{1_0}$ does not depend on $e$ and $L_{p_1}$ coincides with the equilibrium of the Maxwell's Ring R(N+1)BP with equal masses;*
3. *there exists an admissible value of $e$, such that the equilibrium point $L_{p_1} = (\bar{x}_{1_0}, 0, 0)$ for all $\beta > 0$.*

*Proof.* When $e \to 0$, we can write the equation $f_1(x) = f_2(x)$ as

$$\frac{\rho(\Lambda + \beta\rho^2)}{\beta} x^3 = 1 + \frac{1}{\beta} x^2 \sum_{i=1}^{n} \frac{(x - x_i)}{((x - x_i)^2 + y_i^2)^{3/2}},$$

which clearly has one solution for $x > 1/\rho$ (using Lemma 6.5).

When $\beta \to 0$ the equation $f_1(x) = f_2(x)$ transforms into

$$\rho\Lambda x^3 = x^2 \sum_{i=1}^{n} \frac{(x - x_i)}{((x - x_i)^2 + y_i^2)^{3/2}}.$$

Thus, the equilibrium point $(\bar{x}_{1_0}, 0, 0)$ coincides with the equilibrium of the restricted $(N+1)$-body problem (see [9], case $m_0 = 0$, the authors called him $R_1$).

For the last statement, recall that $\bar{x}_1$ is the only positive solution of the equation (29). This equation can be written as

$$x^2 \left( \rho\Lambda x - \sum_{i=1}^{n} \frac{(x - x_i)}{((x - x_i)^2 + y_i^2)^{3/2}} \right) + \frac{\beta}{x} \left( \rho^4 \left( \frac{1}{\rho} - 2e \right) x^4 - x + 2e \right) = 0.$$

Substituting $x = \bar{x}_{1_0}$ in the above equation, the first term vanishes and we get that

$$\rho^4 \left( \frac{1}{\rho} - 2e \right) \bar{x}_{1_0}^4 - \bar{x}_{1_0} + 2e = 0.$$

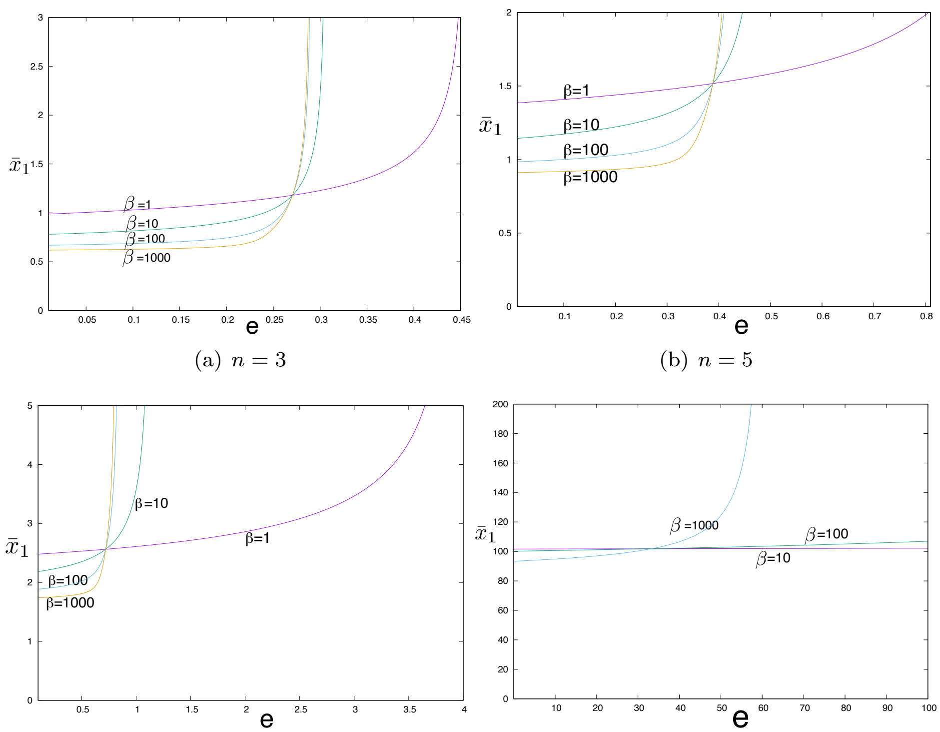
Solving for $e$,

$$e = \frac{\bar{x}_{1_0}(\rho^2 \bar{x}_{1_0}^2 + \rho \bar{x}_{1_0} + 1)}{2(1+\rho \bar{x}_{1_0})(1+\rho^2 \bar{x}_{1_0}^2)} < \frac{1}{2\rho},$$

which is an admissible value.

□

In Figure 8, we show the variation of the coordinate $\bar{x}_1$ of the equilibrium point $L_{p_1}$ for several values of $\beta$ and $n$. We can see the intersection point $\bar{x}_{1_0}$. The approximate value of $e$ for which $L_{p_1} = (\bar{x}_{1_0}, 0, 0)$ for some values of $n$ are given in Table 1.




(a) $n = 3$ (b) $n = 5$

(c) $n = 10$ (d) $n = 500$

FIGURE 8. Variation of the coordinate $\bar{x}_1$ of the equilibrium point $L_{p_1}$ as a function of $e$ for $n = 3, 5, 10, 500$.

| $n$ | $\bar{x}_{1_0}$ | $e$ |
|---|---|---|
| 3 | 1.1799984049 | 0.27099478169 |
| 5 | 1.4548950111 | 0.36616775409 |
| 10 | 2.5629997052 | 0.50888405339 |
| 500 | 101.8255392116 | 0.59920105662 |

TABLE 1. The approximate value of $e$ for which $\bar{x}_1 = \bar{x}_{1_0}$, see Proposition 3.2.

Next result states the number of equilibrium points along $\mathcal{R}_2$, that is, when $0 < x < 1/\rho$ at the left hand side of the peripheral.

**Theorem 3.4.** *For any $\beta > 0$, there exists a value $e^* = e^*(\beta) > 0$ such that the number of equilibrium points along the $\mathcal{R}_2$ is*

(1) 0 *if* $e \in (e^*, e_0)$,
(2) 1 *if* $e = e^*$,
(3) 2 *if* $0 < e < e^*$,
(4) 1 *if* $e \leq 0$.

*Furthermore, $e^* < 3e_0/4$, where $e_0$ is given in (11).*

*Proof.* We are looking for solutions of $f_1(x) = f_2(x)$ for $0 < x < 1/\rho$.

First, we consider $0 < e < e_0$. On one hand, from Lemma 6.4, $f_1$ has a unique minimum at $x^* = x^*(e)$, and $x^*(e) > 1/\rho$ and $f_1(x^*(e)) > 0$ for $e > 3e_0/4$. Using Lemma 6.5, $f_2$ is negative, so for $e > 3e_0/4$ the two functions do not intersect.

On the other hand, also using Lemma 6.4, $\lim_{e\to 0} f_1(x^*(e)) = -\beta$, so that for small values of $e$, $f_1(x^*) < f_2(x^*)$ and the two functions intersect twice. Therefore, by continuity, there exists a value of $e$ such that $f_1$ and $f_2$ coincide tangentially only once for $0 < x < 1/\rho$.

In the case $e \leq 0$, again from Lemmas 6.4 and 6.5, $f_1$ is an increasing function from $-\infty$ or $-n/\rho^2$ when $e < 0$, or $e = 0$, respectively at $x = 0$, to $\infty$ at $x$ tend to $\infty$ and $f_2$ decreases from $f_2(0) = 0$ to $-\infty$ at $x = 1/\rho$. Clearly, $f_1$ and $f_2$ intersect at only one point. □

**Definition 3.4.** *For the values of $e \in (-\infty, e^*]$ we denote the equilibrium points $L_{p_i} = (\bar{x}_i, 0, 0)$, $i = 2, 3$, where $0 < \bar{x}_3 \leq \bar{x}_2 < 1/\rho$, and the equality holds when $e = e^*$ or $e \leq 0$.*

From the proof of Theorem 3.4 it follows easily the next result.

**Proposition 3.3.** *For any fixed value of $n$, for any $\beta$, let $e^*$ and $x^*$ be as in Theorem 3.4 and Definition 3.3, respectively. Then, for any $e < e^*$, the equilibrium point $L_{p_3}$ satisfies that $0 < \bar{x}_3 < x^*$.*

In Figure 9 we can see the variation of $e^*$ for different values of $n$ (left) and the regions where there are 0 and 2 equilibria, and the bifurcation curve $e = e^*$, where there is only one equilibrium (right).

### 3.3. Equilibrium points on half line $\mathcal{L}$.

We will use complex coordinates to write the equilibrium points on the straight line $y = \tan(\pi/n)x$, that is $L_m = re^{i\pi/n}$. From the first two equations of (23) taking $x = r\cos(\pi/n)$ and $y = r\sin(\pi/n)$, multiplying the second equation by the imaginary unit, then adding the two equations, the imaginary part vanishes and the real part gives the equation

$$\Delta r^3 - \beta + \frac{2e\beta}{r} - \sum_{j=1}^{n} \frac{1 - \frac{1}{\rho r}\cos(\frac{2\pi j}{n} + \frac{\pi}{n})}{(1 + \frac{1}{(\rho r)^2} - \frac{2}{\rho r}\cos(\frac{2\pi j}{n} + \frac{\pi}{n}))^{3/2}} = 0. \quad (32)$$

**Theorem 3.5.** *For any fixed value of $n$, for any $\beta > 0$ and admissible $e$, consider the half line $\mathcal{L}$ and $\mathcal{C}$ the circumference containing the peripherals.*

(1) *If $0 < e < e_0$, there exist at least two equilibrium points on $\mathcal{L}$. One of them is inside the circumference $\mathcal{C}$ and the other is outside of $\mathcal{C}$.*
(2) *If $e \leq 0$, there exists at least one equilibrium point on $\mathcal{L}$. It is outside the circumference $\mathcal{C}$.*

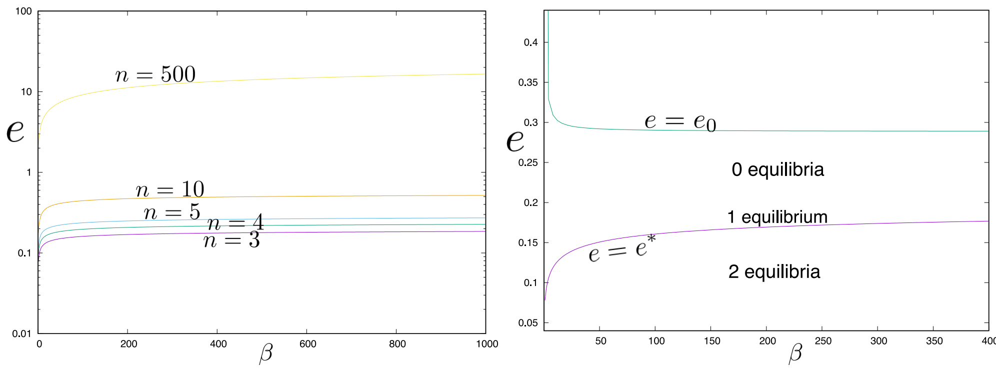



FIGURE 9. Left: variation of the function $e = e^*(\beta)$, for different values of $n$, see Theorem 3.4. Right: regions in the $(\beta, e)$-plane with different number of equilibrium points for $n = 3$.

*Proof.* The proof follows directly from Lemma 6.6. In the case $e > 0$, the left-hand side function from equation (32) has a parabolic behaviour with a negative value at $r = 1/\rho$, so at least must have two zeros, each one at each side of $\mathcal{C}$.

For $e \leq 0$, the function has one change of sign with a negative value at $r = 1/\rho$, so at least has one zero for $r > 1/\rho$. □

**Definition 3.5.** *In the case $0 < e < e_0$, we denote the equilibrium points $L_{m_1}$ and $L_{m_2}$ located outside and inside $\mathcal{C}$, respectively.*

## 4. Linear Stability of the equilibrium solutions

We study the linear stability of the equilibrium points $L_z$ and $L_\xi$, $\xi \in \{p_j, m_k\}$, $j = 1, 2, 3$, $k = 1, 2$, through the analysis of the eigenvalues of the differential matrix of the vector field of the system (15), given by:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 + V_{xx} & V_{xy} & V_{xz} & 0 & 2 & 0 \\ V_{xy} & 1 + V_{yy} & V_{yz} & -2 & 0 & 0 \\ V_{xz} & V_{yz} & V_{zz} & 0 & 0 & 0 \end{pmatrix}, \tag{33}$$

where

$$(34)\quad \begin{aligned}
V_{xx} &= \frac{-1}{\Delta}\left[\beta\left(\frac{1}{r_0^3}-\frac{2e}{r_0^4}\right)+\sum_{j=1}^{n}\frac{1}{r_j^3}\right]+\frac{1}{\Delta}\left[\beta\left(\frac{3}{r_0^5}-\frac{8e}{r_0^6}\right)x^2+3\sum_{j=1}^{n}\frac{(x-x_j)^2}{r_j^5}\right],\\
V_{xy} &= \frac{1}{\Delta}\left[\beta\left(\frac{3}{r_0^5}-\frac{8e}{r_0^6}\right)xy+3\sum_{j=1}^{n}\frac{(x-x_j)(y-y_j)}{r_j^5}\right],\\
V_{xz} &= \frac{z}{\Delta}\left[\beta\left(\frac{3}{r_0^5}-\frac{8e}{r_0^6}\right)x+3\sum_{j=1}^{n}\frac{x-x_j}{r_j^5}\right],\\
V_{yy} &= \frac{-1}{\Delta}\left[\beta\left(\frac{1}{r_0^3}-\frac{2e}{r_0^4}\right)+\sum_{j=1}^{n}\frac{1}{r_j^3}\right]+\frac{1}{\Delta}\left[\beta\left(\frac{3}{r_0^5}-\frac{8e}{r_0^6}\right)y^2+3\sum_{j=1}^{n}\frac{(y-y_j)^2}{r_j^5}\right],\\
V_{yz} &= \frac{z}{\Delta}\left[\beta\left(\frac{3}{r_0^5}-\frac{8e}{r_0^6}\right)y+3\sum_{j=1}^{n}\frac{y-y_j}{r_j^5}\right],\\
V_{zz} &= \frac{-1}{\Delta}\left[\beta\left(\frac{1}{r_0^3}-\frac{2e}{r_0^4}\right)+\sum_{j=1}^{n}\frac{1}{r_j^3}\right]+\frac{z^2}{\Delta}\left[\beta\left(\frac{3}{r_0^5}-\frac{8e}{r_0^6}\right)+3\sum_{j=1}^{n}\frac{1}{r_j^5}\right].
\end{aligned}$$

Since $V_{xz}=V_{yz}=0$ for all the equilibrium points, we can separate the planar and the vertical stability. The characteristic polynomial associated to matrix (33) is

$$(35)\qquad p(\lambda)=(\lambda^2-V_{zz})\bar{p}(\lambda),$$

where

$$(36)\qquad \bar{p}(\lambda)=\lambda^4+(2-V_{xx}-V_{yy})\lambda^2+1+V_{xx}+V_{yy}+V_{xx}V_{yy}-V_{xy}^2.$$

Therefore, the vertical stability of all the equilibrium points is given by the eigenvalues

$$(37)\qquad \pm\lambda_3=\pm\sqrt{V_{zz}},$$

and the planar stability is given by the solutions of $\bar{p}(\lambda)=0$.

Next we study separately the point $L_z$ and the planar equilibria.

4.1. **Stability of the equilibrium point $L_z$.** Consider the equilibrium point $L_z=(0,0,\bar{z})$, given in Theorem 3.2. Using the fact that $\bar{z}$ must satisfy the relation (28), it is not difficult to see that

$$(38)\quad \begin{aligned}
&V_{xy}(L_z)=V_{xz}(L_z)=V_{yz}(L_z)=0,\\
&V_{xx}(L_z)=V_{yy}(L_z)=\frac{3\beta(2e-\bar{z})}{2\rho^2\Delta\left(\bar{z}^2+\frac{1}{\rho^2}\right)\bar{z}^4},\\
&V_{zz}(L_z)=\frac{\beta}{\Delta(\rho^2+\bar{z}^2)z^4}\left(3\bar{z}-8e-2e\rho^2\bar{z}^2\right).
\end{aligned}$$

**Proposition 4.1.** *For each integer $n\geq 2$, $\beta>0$ and an admissible $e>0$, the eigenvalues associated to the the equilibrium point $L_z$ are $\pm\lambda_3=\pm w\mathrm{i}$, $w>0$, and $\lambda_1=a+b\mathrm{i}$, $\bar{\lambda}_1$, $-\lambda_1$, $-\bar{\lambda}_1$, $a>0,\ b>0$.*

*Proof.* Using (35), (37) and (38) the eigenvalues of the matrix in (33) are $\pm\lambda_3=\pm\sqrt{V_{zz}(L_z)}$ and the solutions of

$$\bar{p}(\lambda)=\lambda^4-(2\gamma-2)\lambda^2+(1+\gamma)^2,$$

where $\gamma = V_{xx}(L_z) > 0$.

On one hand, using the fact that $\bar{z} < 2e$ (see Theorem 3.2), we have that

$$3\bar{z} - 2e\rho^2\bar{z}^2 - 8e < 3\bar{z} - 8e < 6e - 8e = -2e < 0.$$

Therefore, $V_{zz}(L_z) < 0$ and two of the eigenvalues are pure imaginary.

On the other hand, the solutions of $\bar{p}(\lambda) = 0$ are

$$\lambda_{\pm}^2 = \gamma - 1 \pm 2\mathrm{i}\sqrt{\gamma}.$$

This completes the proof. □

Therefore, the equilibrium point $L_z$ is of type center × complex saddle, and it is unstable.

4.2. **Stability of planar equilibrium points.** Consider the equilibrium points $L_\xi$, $\xi \in \{p_j, m_k\}$, $j = 1, 2, 3$ and $k = 1, 2$ (see Theorems 3.3, 3.4 and 3.5 respectively). Recall that

$$V_{xz}(L_\xi) = V_{yz}(L_\xi) = 0,$$

and

$$V_{zz}(L_\xi) = -\frac{1}{\Delta}\left[\beta\left(\frac{1}{r_0^3} - \frac{2e}{r_0^4}\right) + \sum_{j=1}^{n}\frac{1}{r_j^3}\right].$$

As we have seen, we can study separately the vertical stability and the planar stability. Using (37), for the vertical stability it is enough to study the sign of $V_{zz}(L_\xi)$.

**Lemma 4.1.** *For each integer $n \geq 2$, $\beta > 0$ and an admissible $e$, $V_{zz}(L_\xi) < 0$.*

*Proof.* When $e \leq 0$ it is clear that $V_{zz}(L_\xi) < 0$.

Consider now $0 < e < e_0$. For $L_{p_i} = (\bar{x}_i, 0, 0)$, we use the equations of the equilibrium points (29) to write

$$V_{zz}(L_{p_i}) = -\frac{1}{\Delta}\left(\Delta + \frac{1}{\rho\bar{x}_i}\sum_{j=1}^{n}\frac{\cos(\frac{2\pi j}{n})}{(\bar{x}_i^2 + \frac{1}{\rho^2} - \frac{2\bar{x}_i}{\rho}\cos(\frac{2\pi j}{n}))^{3/2}}\right). \tag{39}$$

Now using Lemma 6.3, the sum in the above equation is positive and the proof is completed.

For $L_{m_k}$, we use the equations of the equilibrium points (32). For each $k$, we write $L_{m_k} = re^{\mathrm{i}\frac{\pi}{n}}$ (being $r$ different for each $k$). Then we have that

$$\begin{aligned} V_{zz}(L_{m_j}) &= -\frac{1}{\Delta}\left[\left(\frac{1}{r^3} - \frac{2e}{r^4}\right)\beta + \sum_{j=1}^{n}\frac{1}{(r^2 + \frac{1}{\rho^2} - \frac{2r}{\rho}\cos(\frac{2\pi j}{n} + \frac{\pi}{n}))^{3/2}}\right] \\ &= -\frac{1}{\Delta}\left(\Delta + \frac{1}{\rho r}\sum_{j=1}^{n}\frac{\cos(\frac{2\pi j}{n} + \frac{\pi}{n})}{(r^2 + \frac{1}{\rho^2} - \frac{2r}{\rho}\cos(\frac{2\pi j}{n} + \frac{\pi}{n}))^{3/2}}\right). \end{aligned} \tag{40}$$

Let $r' = r\rho$ and

$$g(r') = \sum_{j=1}^{n}\frac{\cos(\frac{2\pi j}{n} + \frac{\pi}{n})}{((r')^2 + 1 - 2r'\cos(\frac{2\pi j}{n} + \frac{\pi}{n}))^{3/2}}. \tag{41}$$

Lemma 3 in [8] (Bang and Elmabsout, 2003) the authors proved that $g$ can be rewritten as an integral of a positive continuous function for $0 < r' < 1$. Using that $g(1/r') = (r')^3 g(r')$, we also obtain that $g(r') > 0$, when $r' > 1$. Thus, $V_{zz}(L_{m_j}) < 0$.

□

Therefore, the eigenvalues $\pm\lambda_3$ associated to the vertical stability of the equilibrium points $L_{p_i}$ and $L_{m_j}$, with $i = 1, 2, 3$ and $j = 1, 2$ are pure imaginary.

To study of the planar stability we will use the same technique introduced by Bang and Elmabsout in [9]. For ease of reading we will use notations similar to them.

We write the polynomial $\bar{p}$ in (36) as

$$\bar{p}(\lambda) = \lambda^4 + 2(1-A)\lambda^2 + (A+1)^2 - |B|^2,$$

where $A = \frac{1}{2}(V_{xx} + V_{yy})$ and $(A+1)^2 - |B|^2 = V_{xx}V_{yy} + V_{xx} - V_{xy}^2 + V_{yy} + 1$. Note that the eigenvalues of the linearized system will be pure imaginary, if and only if, the roots of the previous polynomial are non-positive. This condition is equivalent to

$$\begin{aligned} l_1 &= |B|^2 - 4A > 0, \\ l_2 &= 1 - A > 0, \\ l_3 &= 1 + A - |B| > 0. \end{aligned} \tag{42}$$

Next, we separate the study for the equilibria on the $x$-axis, $L_{p_i}$ and the equilibria on the bisector $L_{m_k}$.

4.2.1. *Planar stability of the equilibrium points $L_{p_j}$.* We consider the points $L_{p_j}$, $j = 1, 2, 3$. From (29), we have that

$$\bar{x}_j - \frac{1}{\Delta}\left(\beta\left(\frac{1}{\bar{x}_j^2} - \frac{2e}{\bar{x}_j^3}\right) + \sum_{l=1}^{n} \frac{\bar{x}_j - x_l}{((\bar{x}_j - x_l)^2 + y_l^2)^{3/2}}\right) = 0.$$

As in [9] we will write $A$ and $B$ in complex coordinates, that is,

$$\begin{aligned} A &= \frac{1}{2\Delta}\sum_{j=1}^{n} \frac{1}{|w_0 - \omega_j|^3} + \frac{\beta}{2\Delta}\frac{1}{|w_0|^3} - \frac{2e\beta}{\Delta}\frac{1}{|w_0|^4} \\ B &= \frac{3}{2\Delta}\sum_{j=1}^{n} \frac{1}{|w_0 - \omega_j|^3}\frac{w_0 - \omega_j}{\overline{w_0 - \omega_j}} + \frac{3\beta}{2\Delta}\frac{1}{|w_0|^3}\frac{w_0}{\overline{w_0}} - \frac{4e\beta}{\Delta}\frac{1}{|w_0|^4}\frac{w_0}{\overline{w_0}}, \end{aligned} \tag{43}$$

with $w_0 = \bar{x}_l$, $l = 1, 2, 3$, $\omega_j = \frac{1}{\rho}e^{i\varphi_j}$ and $\varphi_j = 2\pi j/n$, $j = 1, \ldots, n$.

**Lemma 4.2.** *For each $\beta > 0$,*

1. *If $e < e_0$ and $x = w_0 \in (\frac{1}{\rho}, +\infty)$ (equilibrium solution), $B(x = \omega_0) = |B(x = \omega_0)|$.*
2. *If $e \leq 0$ or $e \to 0^+$ and $x = w_0 \in (0, \frac{1}{\rho})$, $B(x = \omega_0) = |B(x = \omega_0)|$.*

*Proof.* $B(x)$ admits a symmetry when changing $x \to 1/x$, so we can assume $x = \frac{1}{\rho s}$. Thus, $B(x) = \frac{3}{2\Delta}\sum_{j=1}^{n}\frac{1}{|x-\omega_j|^3}\frac{x-\omega_j}{\overline{x-\omega_j}} + \frac{3\beta}{2\Delta}\frac{1}{|x|^3} - \frac{4e\beta}{\Delta}\frac{1}{|x|^4}\frac{x}{\overline{x}}$ which is equivalent to

$$B\left(\frac{1}{\rho s}\right) = \frac{3\rho^3 s^3}{2\Delta}\sum_{j=1}^{n}\frac{1 - s\bar{\omega}_j}{1 - s\bar{\omega}_{-j}}\left(1 + s^2 - 2s\cos\left(\frac{2\pi j}{n}\right)\right)^{-3/2} + \frac{3\beta\rho^3}{2\Delta}s^3 - \frac{4e\beta\rho^4}{\Delta}s^4,$$

with $\bar{\omega}_j = e^{\frac{i\ 2\pi j}{n}}$. We introduce the notation

$$\{f(u)\}_n = \frac{1}{n}\sum_{j=1}^{n} f\left(\frac{2\pi j}{n}\right)$$

$$B\left(\frac{1}{\rho s}\right) = \frac{3n\rho^3 s^3}{2\Delta}\left\{\frac{1 - se^{iu}}{(1 - se^{-iu})(1 - se^{iu})^{3/2}(1 - se^{-iu})^{3/2}}\right\}_n + \frac{3\beta\rho^3}{2\Delta}s^3 - \frac{4e\beta\rho^4}{\Delta}s^4$$

Note that the equation (29) (using $s = 1/(\rho x)$) is equivalent to

$$s^3 = 2e\rho s^4 + \frac{\Delta}{\beta\rho^3} - \frac{n}{\beta}s^3 h_n(s), \tag{44}$$

then

$$\begin{aligned} B &= \frac{3n\rho^3 s^3}{2\Delta}\left\{\frac{1-se^{iu}}{(1-se^{-iu})(1-se^{iu})^{3/2}(1-se^{-iu})^{3/2}}\right\}_n + \tfrac{3}{2} - \tfrac{4e\beta\rho^4}{\Delta}s^4 - \tfrac{3n}{2\Delta}s^3 h_n(s) \\ &= \frac{3n\rho^3 s^3}{2\Delta}\left[\left\{\frac{1-se^{iu}}{(1-se^{-iu})(1-se^{iu})^{3/2}(1-se^{-iu})^{3/2}}\right\}_n - h_n(s)\right] + \tfrac{3}{2} - \tfrac{4e\beta\rho^4}{\Delta}s^4. \end{aligned}$$

Let $B_1 = \left\{\dfrac{1-se^{iu}}{(1-se^{-iu})(1-se^{iu})^{3/2}(1-se^{-iu})^{3/2}}\right\}_n - h_n(s)$,

$$\begin{aligned} B_1 &= \left\{\frac{1-se^{iu}}{(1-se^{-iu})(1-se^{iu})^{3/2}(1-se^{-iu})^{3/2}} - \frac{1-se^{iu}}{(1-se^{iu})^{3/2}(1-se^{-iu})^{3/2}}\right\}_n \\ &= s\left\{\frac{e^{-iu}(1-se^{iu})}{(1-se^{-iu})(1-se^{iu})^{3/2}(1-se^{-iu})^{3/2}}\right\}_n \\ &= s\left\{e^{-iu}\frac{1}{(1-se^{-iu})^2(1-se^{-iu})^{1/2}(1-se^{iu})^{1/2}}\right\}_n \end{aligned}$$

Using the expansion $\frac{1}{(1-z)^{1/2}} = \sum_{k=1}^{\infty} a_k z^k$, with $a_k > 0$, we obtain then

$$B_1 = s\left\{e^{-iu}\sum_{k=0}^{\infty}(k+1)s^k e^{-iku}\sum_{k=0}^{\infty}a_k s^k e^{-iku}\sum_{k=0}^{\infty}a_k s^k e^{iku}\right\}_n = \left\{\sum_{p=0}^{\infty}\{P(e^{iu}, e^{-iu})s^p\}_n\right\},$$

where $P(e^{iu}, e^{-iu})$ is polynomial with positive coefficients. Then $B_1 > 0$, thus $B > 0$, when $0 < s < 1$, for all $0 < e < e_0$. In the case when $e < 0$, we consider, again

$$B(x) = \frac{3n\rho^3 s^3}{2\Delta}\left\{\frac{1-se^{iu}}{(1-se^{-iu})(1-se^{iu})^{3/2}(1-se^{-iu})^{3/2}}\right\}_n + \frac{3\beta\rho^3}{2\Delta}s^3 - \frac{4e\beta\rho^4}{\Delta}s^4,$$

where the term

$$\left\{\frac{1-se^{iu}}{(1-se^{-iu})(1-se^{iu})^{3/2}(1-se^{-iu})^{3/2}}\right\}_n$$

is positive, the proof is similar to the proof $B_1 > 0$. Then $B > 0$, when $s \in (0, +\infty) - \{1\}$ and $e < 0$. Note that if $e \to 0^+$, $B > 0$, thus, for continuity on $e$, we have that for values of $e$ close to 0, $B > 0$, when $s > 1$. □

The following technical lemma will be used later, the proof can be seen in [9].

**Lemma 4.3.** *For every* $s \in (0, +\infty) - \{1\}$,

$$\left\{3\frac{1+s^2e^{2iu}-2se^{iu}}{(1+s^2-2s\cos u)^{5/2}} - \frac{1}{(1+s^2-2s\cos u)^{3/2}} - 2\frac{1-se^{iu}}{(1+s^2-2s\cos u)^{3/2}}\right\}_n > 0$$

Now we see what is the stability characteristic of the equilibrium point $L_{p_1}$.

**Proposition 4.2.** *For each* $\beta$ *and* $e$ *admissible,* $L_{p_1}$ *is unstable.*

*Proof.* Using Lemma 4.2, that is, $|B| = B$ and the equation (44), that is,

$$1 = \frac{\beta}{\Delta}s^3 + \frac{ns^3}{\Delta}h_n(s) - \frac{2\beta e}{\Delta}s^4,$$

then

$$\begin{aligned} |B| - A - 1 &= B - A - 1 \\ &= \frac{n\rho^3 s^3}{2\Delta}\left\{3\frac{1+s^2e^{2iu}-2se^{iu}}{(1+s^2-2s\cos u)^{5/2}} - \frac{1}{(1+s^2-2s\cos u)^{3/2}} - 2\frac{1-se^{iu}}{(1+s^2-2s\cos u)^{3/2}}\right\}_n \\ &+\frac{3\beta\rho^3}{2\Delta}s^3 - \frac{4e\beta\rho^4}{\Delta}s^4 - \frac{\beta\rho^3}{2\Delta}s^3 + \frac{2e\beta\rho^4}{\Delta}s^4 - \frac{\beta}{\Delta\rho^3}s^3 + \frac{2\beta e\rho^4}{\Delta}s^4 \\ &= \frac{n\rho^3 s^3}{2\Delta}\left\{3\frac{1+s^2e^{2iu}-2se^{iu}}{(1+s^2-2s\cos u)^{5/2}} - \frac{1}{(1+s^2-2s\cos u)^{3/2}} - 2\frac{1-se^{iu}}{(1+s^2-2s\cos u)^{3/2}}\right\}_n \end{aligned}$$

Now, using Lemma 4.3 is obtained that $|B| - A - 1 > 0$, thus $l_3 < 0$. Therefore there must be a root of the characteristic polynomial $\bar{p}(\lambda)$ with a non-zero real part. Thus, $L_{p_1}$ is unstable. □

**Proposition 4.3.** *For each $\beta$ and $e \le 0$, $L_{p_2}$ is unstable.*

*Proof.* Using Lemma 4.2 and Lemma 4.3 the result is obtained in a similar form as in Proposition 4.2. □

**Proposition 4.4.** *For each $\beta > 0$ and $0 < e < e^* < \frac{3e_0}{4}$, with $e^*$ bifurcation parameter (as in Theorem 3.4), $L_{p_2}$ ($x_2 \in (x^*, 1/\rho)$, with $x^*$ as in Lemma 6.4) is unstable.*

*Proof.* Remember, from Lemma 4.2

$$B = \frac{3n\rho^3 s^3}{2\Delta}s\left\{e^{-iu}\frac{1}{(1-se^{-iu})^2(1-se^{-iu})^{1/2}(1-se^{iu})^{1/2}}\right\}_n + \frac{3}{2} - \frac{4e\beta\rho^4}{\Delta}s^4.$$

Notice that $x^* < x < 1/\rho$ is equivalent to $1 < s < s^*$, with $s^* = 1/(\rho x^*) = (3\Delta/(2\beta e))^{1/4}/\rho$, then $\frac{3}{2} - \frac{4e\beta\rho^4}{\Delta}s^4 > 0$. Thus, $B > 0$. To prove that $l_3 < 0$, proceed in a similar way to the proof of Proposition 4.2. □

4.2.2. *Planar stability of the equilibrium points $L_{m_j}$.* The equilibrium points on the straight line $y = \tan(\pi/n)x$ in the complex variable are of the form $L_{m_j} = re^{i\pi/n}$, with $j = 1, 2$. Recall that these equilibrium points satisfies the equation

$$\Delta r^3 - \beta + \frac{2e\beta}{r} - \sum_{j=1}^{n}\frac{1 - \frac{1}{\rho r}\cos(\frac{2\pi j}{n} + \frac{\pi}{n})}{(1 + \frac{1}{(\rho r)^2} - \frac{2}{\rho r}\cos(\frac{2\pi j}{n} + \frac{\pi}{n}))^{3/2}} = 0. \tag{45}$$

Note that equation (45) (using $s = \frac{1}{\rho r}$) is equivalent to

$$\frac{\Lambda}{\rho^2} + \beta - 2\beta e\rho - \beta s^3 + 2e\beta\rho s^4 - s^3 h_n(s, \pi/n) = 0, \tag{46}$$

with $h_n(s, \pi/n) = \sum_{j=1}^{n}\dfrac{1 - s\cos\left(\frac{2\pi j}{n} + \frac{\pi}{n}\right)}{(1 + s^2 - 2s\cos\left(\frac{2\pi j}{n} + \frac{\pi}{n}\right))^{3/2}}$. If we divide the equation (46) by $\beta$ and make $\beta$ tend to infinity, it is clear that for large $\beta$, $s$ tends to 1 or $s$ tend to $\bar{s}$, where $\bar{s}$ satisfies the equation $2\rho e - 1 + (2\rho e - 1)s + (2\rho e - 1)s^2 + 2\rho e s^3 = 0$, the second case happens only if $e > 0$. From the equation (46) is obtained

$$\beta = \frac{s^3 h_n(\pi/n, s) - \frac{\Lambda}{\rho^2}}{1 - s^3 - 2e\rho(1 - s^4)}, \tag{47}$$

and

$$1 = \frac{\rho^3}{\Delta}(\beta s^3 + s^3 h_n(s, \pi/n) - 2\beta e \rho s^4). \tag{48}$$

Equations (47) and (48) will be used later.

To study planar linear stability, we can use what was seen in the previous section, that is, we can analyse the values $l_1$, $l_2$, $l_3$ over the equilibria $L_{m_i}$. For this, we must calculate $A$ and $B$ defined in the previous section.

$$\begin{aligned}
A &= \frac{1}{2\Delta}\sum_{j=1}^{n}\frac{1}{|w_0-\omega_j|^3} + \frac{\beta}{2\Delta}\frac{1}{|w_0|^3} - \frac{2e\beta}{\Delta}\frac{1}{|w_0|^4} \\
&= \frac{\rho^3 s^3}{2\Delta}\sum_{j=1}^{n}\frac{1}{\left(1+s^2-2s\cos\left(\frac{2\pi j}{n}+\frac{\pi}{n}\right)\right)^{3/2}} + \frac{\beta\rho^3}{2\Delta}s^3 - \frac{2e\beta\rho^4}{\Delta}s^4 \\
&= \frac{\rho^3 s^3}{2\Delta(1-s^3-2e\rho(1-s^4))} \\
&\times\left(\sum_{j=1}^{n}\frac{1-s^4\cos(\frac{2\pi j}{n}+\frac{\pi}{n})-2e\rho(1-s^4)-4e\rho s^4(1-s\cos(\frac{2\pi j}{n}+\frac{\pi}{n}))}{(1+s^2-2s\cos(\frac{2\pi j}{n}+\frac{\pi}{n}))^{3/2}} - (1-4\rho e s)\frac{\Lambda}{\rho^2}\right), \\
B &= \frac{3}{2\Delta}\sum_{j=1}^{n}\frac{1}{|w_0-\omega_j|^3}\frac{w_0-\omega_j}{\overline{w_0-\omega_j}} + \frac{3\beta}{2\Delta}\frac{1}{|w_0|^3}\frac{3}{2\Delta}\sum_{j=1}^{n}\frac{1}{|w_0-\omega_j|^3}\frac{w_0-\omega_j}{\overline{w_0-\omega_j}} \\
&+\frac{3\beta}{2\Delta}\frac{1}{|w_0|^3}\frac{w_0}{\overline{w_0}} - \frac{4e\beta}{\Delta}\frac{1}{|w_0|^4}\frac{w_0}{\overline{w_0}} - \frac{4e\beta}{\Delta}\frac{1}{|w_0|^4}\frac{w_0}{\overline{w_0}} \\
&= \frac{3\rho^3 s^3}{2\Delta}e^{\frac{i2\pi}{n}}\sum_{j=1}^{n}\frac{1-2s\cos(\frac{2\pi j}{n}+\frac{\pi}{n})+s^2\cos(\frac{4\pi j}{n}+\frac{2\pi}{n})}{(1+s^2-2s\cos(\frac{2\pi j}{n}+\frac{\pi}{n}))^{5/2}} + \frac{3\beta\rho^3}{2\Delta}e^{\frac{i2\pi}{n}}s^3 - \frac{4e\beta\rho^4}{\Delta}e^{\frac{i2\pi}{n}}s^4.
\end{aligned}$$

**Proposition 4.5.** *If $\beta$ is sufficiently large and $\frac{1}{8\rho} < e < \frac{3}{8\rho}$, then $L_{m_1}$ is linearly stable.*

*Proof.* Note that if, $s$ tend to $1^-$, then by (47) $\beta$ is sufficiently large, and so $e_0$ tends to $e_0 = \frac{1}{2\rho}$. On the other hand,

$$|B| = \left|\frac{3\rho^3 s^3}{2\Delta}\sum_{j=1}^{n}\frac{1-2s\cos(\frac{2\pi j}{n}+\frac{\pi}{n})+s^2\cos(\frac{4\pi j}{n}+\frac{\pi}{n})}{(1+s^2-2s\cos(\frac{2\pi j}{n}+\frac{\pi}{n}))^{5/2}} + \frac{3\beta\rho^3}{2\Delta}s^3 - \frac{4e\beta\rho^4}{\Delta}s^4\right|.$$

If $s$ tends to $1^-$ and $e < \frac{3}{8\rho}$, then the argument inside $||$ is positive. Substituting relation (47) in $l_2$ and $l_3$, we obtain

$$\begin{aligned}
l_2 &= 1 - A \\
&= \frac{\beta\rho^3 s^3}{2\Delta} + \frac{\rho^3 s^3}{2\Delta}\sum_{j=1}^{n}\frac{1-2s\cos\left(\frac{(2j+1)\pi}{n}\right)}{\left(2-2\cos\left(\frac{(2j+1)\pi}{n}\right)\right)^{3/2}}, \\
l_3 &= A + 1 - |B| \\
&= \frac{\rho^3 s^3}{2\Delta}\sum_{j=1}^{n}\left(\frac{3-2s\cos\left(\frac{(2j+1)\pi}{n}\right)}{(1+s^2-2s\cos\left(\frac{(2j+1)\pi}{n}\right))^{3/2}} - \frac{3-6s\cos\left(\frac{(2j+1)\pi}{n}\right)-3s^2\cos\left(\frac{(4j+2)\pi}{n}\right)}{(1+s^2-2s\cos\left(\frac{(2j+1)\pi}{n}\right))^{5/2}}\right),
\end{aligned}$$

Thus, when $s$ tends to $1^-$

$$l_1 = \frac{3\rho^3}{2\Delta}\sum_{j=1}^{n}\frac{1-2\cos\left(\frac{(2j+1)\pi}{n}\right)+\cos\left(\frac{(4j+2)\pi}{n}\right)}{(2-2\cos\left(\frac{(2j+1)\pi}{n}\right))^{5/2}} + \frac{\beta\rho^3}{\Delta}\left(-\frac{1}{2}+4e\rho\right) > 0,$$

$$l_2 = \frac{\rho^3}{2\Delta}\left(\beta + \sum_{j=1}^{n}\frac{1-2s\cos\left(\frac{(2j+1)\pi}{n}\right)}{\left(2-2\cos\left(\frac{(2j+1)\pi}{n}\right)\right)^{3/2}}\right) > 0,$$

$$l_3\cdot\frac{2\Delta}{\rho^3} = \sum_{j=1}^{n}\frac{\left(\cos\left(\frac{(2j+1)\pi}{n}\right)+3\right)\csc^2\left(\frac{(2j+1)\pi}{2n}\right)}{4\sqrt{2-2\cos\left(\frac{(2j+1)\pi}{n}\right)}} > 0,$$

since $\beta$ is large enough. Finally, $l_1$, $l_2$ and $l_3$ are all positive numbers when $s \to 1^-$ (equivalently $r \to 1/\rho^+$) or when $\beta$ sufficiently large. With these conditions $L_{m_1}$ is linearly stable.

□

## 5. Concluding remarks

We have studied a spatial R$(N+1)$BP where the gravitational attraction of the central body with mass $m_0$ is given by a Manev potential $-1/r + e/r^2$ with $e \neq 0$ and the other $N-1$ bodies of masses equal to $m = \beta m_0$ with Newtonian potential $(-1/r)$, we call this model Spatial Manev R$(N+1)$BP. The problem depends on three parameters, the number of peripherals $n$, the ratio of the mass of the central body to the mass of one of the peripherals, and the Manev parameter, $e$. One of the things we have proven when $e > 0$, is that due to the repulsive term emanating from the central body, it is not possible to have a binary collision between the body of infinitesimal mass and the central body, contrary to the case $e < 0$.

In the present work we focus on studying the existence and stability of equilibrium points. In the first place we have proved that the equilibria exist on the $z$ axis and on the axes of symmetry of the regular polygon formed by the peripherals when $z = 0$. A notable property is that on the z axis, there are two equilibrium points when $e > 0$, both unstable. By the Lyapunov Center Theorem, there exists a family of periodic orbits that lives on the z-axis in that case.

On the lines $\mathcal{R}_j = \{z = 0, y = \tan(2\pi(j-1)/n)x, x > 0\}$, $j = 1, \ldots, n$, there are $2n$ or $3n$ equilibrium points when $n \leq 472$, all unstable independent of the values of the parameters. And on the lines $\mathcal{L}_j = \{z = 0, y = \tan(\pi(2j-1)/n)x, x > 0\}$, $j = 1, \ldots, n$ at least $2n$ equilibria. $n$ of them for some values of $\beta$ and $e$ are linearly stable. The different amounts of equilibria depend on the parameters $\beta$, $e$ and $n$. The results regarding the study of stability, motivate us to study the invariant manifolds of these equilibrium points and the connections between them, in a future work.

## 6. Appendix

The following technical lemma characterizes the roots of a particular type of function (see its proof in [2]).

**Lemma 6.1.** *Let $T$ be a positive constant, $n$ a natural number, and*

$$F(p) = f(p) + \sum_{j=1}^{n-1} f(p+jT) \tag{49}$$

*where $f$ is a function such that*

i) $f(p+nT) = f(p)$,

ii) $f(p) = 0$, *if and only if,* $p = \frac{knT}{2}$, *for all* $k \in \mathbb{Z}$,

iii) $f(-p) = -f(p)$.

*Then* $F(p) = 0$, *if and only if,* $p = \frac{kT}{2}$, $k \in \mathbb{Z}$.

**Lemma 6.2.** *Let*

$$F(x,y,z) = \sum_{i=1}^{n} \frac{y_i}{r_i^3},$$

*where* $r_i^2 = (x-x_i)^2 + (y-y_i)^2 + z^2$, $(x_i, y_i)$ *defined in* (14). *Then, if* $y > 0$, $F(x,y,z) > 0$, *whereas if* $y < 0$ *then* $F(x,y,z) < 0$.

*Proof.* Using $y_i = \sin(\varphi_i)/\rho$, we write $F$ as

$$F(x,y,z) = \frac{1}{\rho} \sum_{i=2}^{\lfloor \frac{n+1}{2} \rfloor} \sin(\varphi_i) \left( \frac{1}{r_i^3} - \frac{1}{r_{n+2-i}^3} \right).$$

Notice that $\sin(\varphi_i) > 0$ for $i = 2, \ldots, \lfloor \frac{n+1}{2} \rfloor$. It is not difficult to see that when $y > 0$ $r_i < r_{n+2-i}$, whereas when $y < 0$, $r_i > r_{n+2-i}$ for all $i = 2, \ldots, \lfloor \frac{n+1}{2} \rfloor$. This concludes the proof. □

**Lemma 6.3.** *Let*

$$F(x,z) = \sum_{i=1}^{n} \frac{x_i}{r_i^3},$$

*where* $r_i^2 = (x-x_i)^2 + y_i^2 + z^2$, $(x_i, y_i)$ *defined in* (14). *Then, if* $x > 0$, $F(x,z) > 0$, *whereas if* $x < 0$ *then* $F(x,z) < 0$.

*Proof.* Using trigonometric identities, for any $\theta$ we have

$$\sin(\theta)\cos(\varphi_i) = \sin(\varphi_i + \theta) - \cos\theta \sin(\varphi_i).$$

Considering $\theta = \frac{2\pi}{n}$, we write $F$ as

$$F(x,z) = \frac{1}{\sin(\theta)} \sum_{i=1}^{n} \frac{\sin(\varphi_{i+1})}{r_i^3} - \cot(\theta) \sum_{i=1}^{n} \frac{\sin(\varphi_i)}{r_i^3}. \tag{50}$$

Note that $\sum_{i=1}^{n} \frac{\sin(\varphi_i)}{r_i^3} = 0$, so the equation (50) is rewritten as

$$F(x,z) = \sum_{i=1}^{n} \frac{\sin(\varphi_{i+1})}{r_i^3} = \frac{1}{\rho \sin(\theta)} \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \sin(\varphi_{i+1}) \left( \frac{1}{r_{n-i}^3} - \frac{1}{r_i^3} \right). \tag{51}$$

If $x > 0$, we claim that $r_i \leq r_{n-i}$ (see Figure 10), for all $i = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$ and $n \geq 3$ (in the case $n = 2$ is evident). In effect,

$$r_i \leq r_{n-i} \quad \Leftrightarrow \quad \cos(\varphi_{n-i}) < \cos(\varphi_i), \;\; \forall i = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor.$$

Note that $\cos(\varphi_{n-i}) = \cos(\varphi_i + \frac{4\pi}{n})$, also $\cos(\varphi_i)$ is a decreasing function in $[0, \pi]$ for all $i = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$, therefore $\cos(\varphi_{n-i}) < \cos(\varphi_i)$, $\forall i = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$ is true. So, $r_i \leq r_{n-i}$, for all $i = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$. Now $\frac{1}{r_{n-i}^3} - \frac{1}{r_i^3} < 0$, therefore $F(x,z) > 0$. On the contrary, if we assume $x < 0$, $r_i \geq r_{n-i}$, for all $i = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$ it is easy to check with the same above argument, then $\frac{1}{r_{n-i}^3} - \frac{1}{r_i^3} > 0$, therefore $F(x,z) < 0$. □

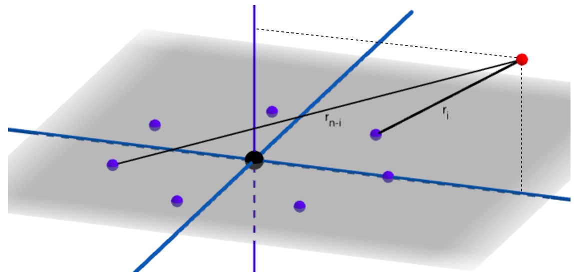



FIGURE 10. Distance between the small particle in position $(x, y, z)$ and the peripheries $P_i$ and $P_{n-i}$, respectively.

Some properties of the function $f_1$, defined in (30) we will resume them in the next Lemma.

**Lemma 6.4.** *For any fixed value of $\beta > 0$ and $e$ admissible, the function $f_1(x)$, $x > 0$, defined in (30) has the following properties:*

(1) *Case $e \leq 0$.*
   (i) *$f_1(x)$ is an increasing function.*
   (ii) $\lim_{x\to+\infty} f_1(x) = +\infty$ *and* $\lim_{x\to 0^+} f_1(x) = -\infty$.
   (iv) *$f_1(0) = -\beta$, when $e = 0$.*

(2) *Case $0 < e < e_0$.*
   (i) *It has only one critical point, which is a minimum, at*

$$x^* = x^*(e) = \left(\frac{2\beta e}{3\Delta}\right)^{1/4}, \tag{52}$$

   *where $e_0$ is given in (11).*
   (ii) *$x^*(e)$ is an increasing function of $e$ and $x^*(3e_0/4) = 1/\rho$.*
   (iii) *$f_1(x^*(e)) = 4\Delta^{1/4}\left(\frac{2\beta e}{3}\right)^{3/4} - \beta$ as a function of $e$ has only one critical point, which is a maximum, at $3e_0/4$.*
   (iv) *$f_1(1/\rho) = \frac{1}{4}\sum_{i=2}^{n} \frac{1}{\sin\left(\frac{\pi(i-1)}{n}\right)} = \frac{\Lambda}{\rho^2}$, is an increasing function in $n$.*

*Proof.* The proof of complete part 1 and the proof of part 2- i), 2- ii) and 2-ii) are straightforward calculations. For part iv), we will just prove that $h(n) = f_1(1/\rho)$ is an increasing function.

Consider the case where $n$ is even; the odd case is similar. Then

$$\begin{aligned} h(n) &= \frac{1}{4}\sum_{j=1}^{n-1} \frac{1}{\sin\left(\frac{\pi j}{n}\right)} = \frac{1}{4}\left(2\sum_{j=1}^{\frac{n}{2}-1} \frac{1}{\sin\left(\frac{\pi j}{n}\right)} + 1\right), \\ h(n+1) &= \frac{1}{4}\sum_{j=1}^{n} \frac{1}{\sin\left(\frac{\pi j}{n+1}\right)} = \frac{1}{4}\left(2\sum_{j=1}^{\frac{n}{2}} \frac{1}{\sin\left(\frac{\pi j}{n+1}\right)}\right). \end{aligned}$$

Now,

$$h(n+1) - h(n) = \frac{1}{4}\left(2\sum_{j=1}^{\frac{n}{2}-1}\left(\frac{1}{\sin\left(\frac{\pi j}{n+1}\right)} - \frac{1}{\sin\left(\frac{\pi j}{n}\right)}\right) + \frac{2}{\sin\left(\frac{n\pi}{2(n+1)}\right)} - 1\right).$$

Since $\frac{1}{\sin\left(\frac{\pi j}{n+1}\right)}-\frac{1}{\sin\left(\frac{\pi j}{n}\right)}>0$, for all $j=1,\dots,\frac{n}{2}-1$ and $\frac{2}{\sin\left(\frac{n\pi}{2(n+1)}\right)}-1>0$, then $h(n+1)-h(n)>0$. Therefore, $h(n)$ is an increasing function in $n$. □

Now, some properties of the function $f_2$, defined in (31), we summarise them in the next Lemma.

**Lemma 6.5.** *The function $f_2(x)$, $x>0$, defined in (31) has the following properties:*

(1) *$f_2(x)>0$, when $x>1/\rho$ and $f_2(x)<0$, when $x\in(0,\frac{1}{\rho})$, and in both cases $f_2$ is decreasing.*

(2) *$\lim_{x\to\infty} f_2(x)=n$, $\lim_{x\to\frac{1}{\rho}^+} f_2(x)=+\infty$, $\lim_{x\to\frac{1}{\rho}^-} f_2(x)=-\infty$ and $\lim_{x\to 0^+} f_2(x)=0$.*

*Proof.* The proof of the item 2 and the first two properties of item 1 are easy to verify, using straightforward calculations. For the last two properties we will use the results of Bang and Elmabsout (2004) in [9] and Moeckel and Simo (1995) in [18]. Now, using that $x_i=\frac{1}{\rho}\cos(\varphi_i)$ and $y_i=\frac{1}{\rho}\sin(\varphi_i)$, with $\varphi_i=\frac{2\pi(i-1)}{n}$, for $i=1,\dots,n$, now

$$f_2(x)=x^2\sum_{i=1}^{n}\frac{x-\frac{1}{\rho}\cos(\varphi_i)}{((x^2-\frac{2x}{\rho}\cos(\varphi_i)+\frac{1}{\rho^2})^{3/2}}=(\rho x)^2\sum_{i=1}^{n}\frac{\rho x-\cos(\varphi)}{((\rho x)^2-2x\rho\cos(\varphi_i)+1)^{3/2}}. \tag{53}$$

Making the change of variable $t=\dfrac{1}{\rho x}$, we obtain

$$f_2(t)=\sum_{i=1}^{n}\frac{1-t\cos(\varphi_i)}{(t^2-2t\cos(\varphi_i)+1)^{3/2}}. \tag{54}$$

We need to see that $f_2(t)$ is increasing for $t\in(0,1)$ and $f_2(t)$ is increasing for $t>1$. In order to [9], we notice that $f(t)=(tV(t))'$, where

$$V(t)=\sum_{i=1}^{n}\frac{1}{(t^2-2t\cos(\varphi_i)+1)^{1/2}}. \tag{55}$$

For $0<t<1$, $V(t)$ is a series in $t$ with all its Taylor coefficients are positive (see [18]). So $V$ and all its derivatives are positive. So, $f_2'(t)=V(t)+tV'(t)>0$. Then $f_2(t)$ is an increasing function for $0<t<1$. Using that $f_2(t)=-\frac{1}{t^2}V'(1/t)$, then $f_2'(t)=\frac{2}{t^3}V'(\frac{1}{t})+\frac{1}{t^4}V''(t)$, it follows that $f_2$ is increasing for $t>1$. □

Some properties of the function $h$ defined in (32) are listed in the next lemma.

**Lemma 6.6.** *For any fixed value of $\beta>0$ and $e$ admissible, the function $h(r)$, defined*

$$h(r)=\Delta r^3-\beta+\frac{2e\beta}{r}-\sum_{j=1}^{n}\frac{1-\frac{1}{\rho r}\cos(\frac{2\pi j}{n}+\frac{\pi}{n})}{(1+\frac{1}{(\rho r)^2}-\frac{2}{\rho r}\cos(\frac{2\pi j}{n}+\frac{\pi}{n}))^{3/2}}$$

*has the following properties.*

(1) *Case $e\le 0$.*
   (i) $\lim_{r\to+\infty} h(r)=+\infty$
   (ii) *$\lim_{r\to 0^+} h(r)=-\infty$, when $e<0$.*
   (iii) *$\lim_{r\to 0^+} h(r)=-\beta$, when $e=0$.*
   (iv) *$h\left(\frac{1}{\rho}\right)<0$.*

(2) *Case $0<e<e_0$.*
   (i) *$\lim_{r\to+\infty} h(r)=+\infty$.*

(ii) $\lim_{r\to 0^+} h(r) = +\infty$.

(iii) $h\left(\frac{1}{\rho}\right) < 0$.

*Proof.* The proof of the case $e \leq 0$ part 1-i), 1-ii) and 1-ii) and the case $e < 0$ and part 2-i), 2-ii) are straightforward calculations. On the other hand,

$$
h\left(\frac{1}{\rho}\right) = \frac{\Lambda}{\rho^2} - \sum_{j=1}^{n} \frac{1-\cos\left(\frac{\pi}{n}+\frac{2\pi j}{n}\right)}{\left(2-2\cos\left(\frac{\pi}{n}+\frac{2\pi j}{n}\right)\right)} = \frac{1}{4}\sum_{j=1}^{n-1}\frac{1}{\sin\left(\frac{\pi j}{n}\right)} - \sum_{j=1}^{n} \frac{1-\cos\left(\frac{\pi}{n}+\frac{2\pi j}{n}\right)}{\left(2-2\cos\left(\frac{\pi}{n}+\frac{2\pi j}{n}\right)\right)}. \tag{56}
$$

Let's define $h_1(n) = \frac{1}{4}\sum_{j=1}^{n-1}\frac{1}{\sin\left(\frac{\pi j}{n}\right)}$ and $h_2(n) = \sum_{j=1}^{n} \frac{1-\cos\left(\frac{\pi}{n}+\frac{2\pi j}{n}\right)}{\left(2-2\cos\left(\frac{\pi}{n}+\frac{2\pi j}{n}\right)\right)}$. Then,

$$
\begin{aligned}
h\left(\frac{1}{\rho}\right) &= h_1(n) - h_2(n) = 2h_1(n) - (h_2(n) + h_1(n)) \\
&= 2(h_1(n) - h_1(2n)) < 0,
\end{aligned}
$$

because $h_1(n)$ is increasing function with respect to $n$ according to Lemma 6.4. □

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