CRITICALITY VIA FIRST ORDER DEVELOPMENT OF THE PERIOD CONSTANTS

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Abstract. In this work we study the criticality of some planar systems of polynomial differential equations having a center for various low degrees n. To this end, we present a method which is equivalent to the use of the first non-identically zero Melnikov function in the problem of limit cycles bifurcation, but adapted to the period function. We prove that the Taylor development of this first order function can be found from the linear terms of the corresponding period constants. Later, we consider families which are isochronous centers being perturbed inside the reversible centers class, and we prove our criticality results by finding the first order Taylor developments of the period constants with respect to the perturbation parameters. In particular, we obtain that at least 22 critical periods bifurcate for $n = 6$, 37 for $n = 8$, 57 for $n = 10$, 80 for $n = 12$, 106 for $n = 14$, and 136 for $n = 16$. Up to our knowledge, these values improve the best current lower bounds.

1. Introduction

Melnikov functions are widely used on the well-known problem of limit cycles bifurcation in planar systems of differential equations, in the line of 16th Hilbert Problem. In analogy to this question, some authors have proposed an equivalent approach for studying the number of oscillations of the period function of a center, also known as critical periods. Works such as [6, 15, 35] propose this technique to deal with the lower bounds on the number of critical periods by using the equivalent to the first order Melnikov function for the period.

Another question intimately related to the periodicity of a system is the isochronicity characterization. Huygens was the forerunner of isochronicity studies and aroused the interest on this line of research, see [3]. In the last 30 years many authors have studied the existence of differential equations with equilibrium points of center type that satisfy this isochronicity property, see for example [11, 22] and the interesting survey of Chavarriga and Sabatini [5].

To deal with the aforementioned problems we will start by introducing some preliminary concepts and classical results on these topics. This introductory part is based on that of our recent work [32], so many of the ideas here presented are directly extracted from that paper.

Let us consider a real polynomial system of differential equations in the plane with a nondegenerate center at the origin, this is the linear part at the equilibrium point having zero trace and positive determinant. It is a well known fact that, by a suitable change of coordinates and time rescaling, it can be written in the form

$$\begin{cases} \dot{x} = -y + X(x,y) =: P(x,y), \\ \dot{y} = x + Y(x,y) =: Q(x,y), \end{cases} \tag{1}$$

where X and Y are polynomials of degree $n \geq 2$ which start at least with quadratic monomials. We define the *period annulus* of a center as the largest neighborhood Ω of

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