NEW LOWER BOUNDS OF THE NUMBER OF CRITICAL PERIODS IN REVERSIBLE CENTERS

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Abstract. In this paper we aim to find the highest number of critical periods in a class of planar systems of polynomial differential equations for fixed degree having a center. We fix our attention to lower bounds of local criticality for low degree planar polynomial centers. The main technique is the study of perturbations of reversible holomorphic (isochronous) centers, inside the reversible centers class. More concretely, we study the Taylor developments of the period constants with respect to the perturbation parameters. First, we see that there are systems of degree $3 \leq n \leq 16$ for which up to first order at least $(n^2+n-4)/2$ critical periods bifurcate from the center. Second, we improve this number for centers with degree from 3 to 9. In particular, we obtain 6 and 10 critical periods for cubic and quartic degree systems, respectively.

1. Introduction

Huygens, with his work on the cycloidal pendulum in the 17th century, was the forerunner of isochronicity studies and aroused the interest of this line of research, see [2]. In the last 30 years many authors have studied the existence of differential equations with equilibrium points of center type that satisfy this isochronicity property, see for example [10, 21] and the interesting survey of Chavarriga and Sabatini [3]. There are other two very related problems, the monotonicity of the period and the bifurcation of critical periods. In this paper we deal with the second one. Before knowing with more detail such problems we need some preliminary concepts definitions and classical results in this research line.

Let us consider a real analytical system of differential equations in the plane with a center at the origin and nonzero linear part. It is a well known fact that, by a suitable change of coordinates and time rescaling, it can be written in the form

$$(\dot{x}, \dot{y}) = (-y + X(x,y), x + Y(x,y)), \tag{1}$$

where X and Y are convergent real series which start at least with quadratic monomials. We define the *period annulus* of a center as the largest neighborhood Ω of the origin with the property that the orbit of every point in $\Omega \setminus \{(0,0)\}$ is a simple closed curve that encloses the origin, so the trajectory of every point in $\Omega \setminus \{(0,0)\}$ is a periodic function. Suppose the origin is a center for system (1) and that the number $\rho^* > 0$ is so small that the segment $\Sigma = \{(x,y) : 0 < x < \rho^*, y = 0\}$ of the x-axis lies wholly within the period annulus. For ρ satisfying $0 < \rho < \rho^*$, let $T(\rho)$ denote the least period of the trajectory through $(x,y) = (\rho, 0) \in \Sigma$. The function $T(\rho)$ is the *period function* of the center, which by the Implicit Function Theorem is real analytic. Moreover, we say that the center of system (1) is isochronous if its period function $T(\rho)$ is constant, which means that every periodic orbit in a neighborhood of the origin has the same period.

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