PERIODIC BOUNCING SOLUTIONS OF THE LAZER-SOLIMINI EQUATION WITH WEAK REPULSIVE SINGULARITY

DAVID ROJAS[1] AND PEDRO J. TORRES[2]

Abstract. We prove the existence and multiplicity of periodic solutions of bouncing type for a second-order differential equation with a weak repulsive singularity. Such solutions can be catalogued according to the minimal period and the number of elastic collisions with the singularity in each period. The proof relies on the Poincaré-Birkhoff Theorem.

1. Introduction

Differential equations with singularities appear as mathematical models in many scientific areas and have been studied from many viewpoints [14]. In this paper, we consider the singular second order differential equation

$$\ddot{u} - \frac{1}{u^\alpha} = p(t), \;\; u > 0, \tag{1}$$

with parameter $\alpha > 0$ and $p : \mathbb{R} \to \mathbb{R}$ a continuous and 2π-periodic function. In a seminal paper, Lazer and Solimini [3] proved that when $\alpha \geqslant 1$ equation (1) has a positive periodic solution if and only if p has negative mean value. The authors also showed that the statement is sharp with respect to the parameter α in the sense that if $0 < \alpha < 1$, a function p with negative mean value can be constructed in such a way that (1) has no periodic solutions. Later, [6, Example 3.9] provided an effective sufficient condition over p for the existence of a classical periodic solution in the weak repulsive case. The particular case $\alpha = 1/2$ has been studied in [8] showing that the equation corresponds to a perturbed isochronous oscillator and resonance conditions on the forcing term $p(t)$ are given.

In the mentioned references, existence of solutions is understood in the classical sense and collisions with the singularity are not allowed. The goal of the present paper is twofold. First, we aim to extend the notion of solutions of equation (1) for $0 < \alpha < 1$ admitting elastic collisions with the singularity at $x = 0$. Second, we prove the existence of harmonic and sub-harmonic bouncing solutions of equation (1) for any negative 2π-periodic forcing $p(t)$.

For the analogous equation with attractive nonlinearity (that is, changing the sign of the second term of the left-hand side of the equation), the notion of bouncing solution has been adequately defined and studied in a number of papers [4, 5, 7, 9, 13, 12, 15]. In contrast, it remains unexplored for the repulsive case. Our aim is to fill, at least partially, this gap.

The structure of the paper is as follows. In Section 2, we analyze in detail the autonomous case (when the forcing term $p(t)$ is constant), including the associated period function and the continuation of colliding orbits. In Section 3, we define rigurously the notion of bouncing solution, proving that the initial boundary value

2010 *Mathematics Subject Classification.* Primary: 34C25.

Key words and phrases. Periodic solution, bouncing, singularity, Poincaré-Birkhoff Theorem.