Qualitative theory of planar piecewise linear differential systems

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To Alba, Montse and Sara.

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Preface

From their beginning the ordinary differential equations (ODEs) have been one of the best language in which scientists from many different fields have shaped the natural phenomena. Currently its use is so extended that some introductory aspects and some elemental methods of solving ODEs are part of the curriculum of the students from various scientific disciplines.

Despite the huge number of natural phenomena that are modeled by ODEs, from these the ones which can be solved explicitly is negligible. The qualitative theory and the numeric simulation are the two main tools that allow us to understand the behaviour of the solutions of there ODEs. The qualitative theory usually works very well locally but only partially when we study the global objects of the nonlinear ODEs.

However few families of ODEs allow a full treatment from the standpoint of the qualitative theory. The family of systems of linear differential equations is one of them. The importance which has this family in the context of the qualitative theory is evidenced when much of the local analysis of nonlinear ODEs is reduced to the study of its linear part. However the richness of the dynamic behaviour of this family is very limited.

With this book we want to emphasize that a first step toward understanding the behaviour of systems of nonlinear ODEs is to consider systems of piecewise linear differential systems (PWLS), which are some of the easiest nonlinear ODEs. More precisely, for the planar PWLS here studied, the full program of the qualitative theory can be applied. Additionally we have the advantage that the richness of their dynamic behaviour is comparable to that of the general nonlinear ODEs.

On the other hand PWLS are important in applications, where they arise in a natural way, for instance in control theory, in electric circuits design, etc. In these disciplines to consider this class of systems is an alternative that fits better to the experiments.

This book is addressed to the mathematicians, engineers and scientifics in general who want to introduce in the qualitative theory of ODEs. It is also indicated as a reference book for anyone who needs to know the global phase portraits and the bifurcation set of all the symmetric three–pieces linear differential systems (here called fundamental systems) because their full characterization appears here by first time.

The book is divided into five chapters. In Chapter 1 we describe the global phase portraits (including their behaviour at infinity) of all the fundamental systems and we characterize all the bifurcations exhibit by these systems when the parameters change.

In Chapter 2 we collect the basic results of the qualitative theory of planar EDOs that will use in the rest of the book. To simplify the exposition of some concepts we have limited the scope of our exposition to ODEs having a complete flow. For this reason some of the results presented here are more restrictive than those that normally appear in the literature. In Section 2.5 we treat the planar linear differential systems. We refer frequently to this section throughout the book. In Section 2.9 we formalize some aspects on the compactification of flows in order to apply this technique to the fundamental systems. The Poincaré compactification is widely used in polynomial differential systems to study the behaviour of the flow near the infinity. However, although some differential equations can be compactified satisfactorily, we have not found a systematization of its use outside of polynomial differential systems.

In Chapter 3 we begin with the study of the fundamental systems and we show that we can apply to them the existence and uniqueness theorem and the continuous dependence on initial conditions and parameters theorem. In this chapter we also prove that the behaviour of these systems is determined by a pair of matrices called fundamental matrices. This justifies that, except in very singular cases, we use the trace and the determinant of both matrices as fundamental parameters to describe the dynamics of these systems. Additionally we study the local phase portrait at the singular points, both finite and infinite, and we give some results about the existence and configuration of the periodic orbits of the fundamental systems.

Poincaré maps of PWLS are determined by the linear differential systems which act in each of the pieces. For fundamental systems, one of these linear differential systems is homogeneous while the other two are non–homogeneous. Consequently, in Chapter 4 we study all the Poincaré maps of linear differential systems associated to two cross sections. These cross sections are parameterized in such away that the Poincaré maps become invariant by linear transformations. This parametrization has important implications. First it allows the study of the Poincaré maps by choosing, in each case, the simplest expression for the fundamental matrices. Usually we will consider that the matrices are expressed in their real Jordan normal form. Second we can characterize the region in the parameter space where we can guarantee the existence of the Poincaré maps. Thus the bifurcation set associated to the existence or not of the Poincaré maps in the parameter space is an algebraic manifold homeomorphic to the Witney umbrella. Finally, this parametrization establishes a link between PWLS having defined the Poincaré maps with differential systems which are called observables in control theory.

By collecting the results obtained in the previous chapters, in Chapter 5 we describe and classify all the phase portraits of the fundamental systems. The description of the phase portraits is performed via the characterization of all separatrices and canonical regions. This allows to use in a rigorous way the Marcus–Newmann–Peixoto Theorem of topological classification of the planar flows. In this chapter we also give expressions of the bifurcation manifolds. Each of the sections of the chapter are devoted to fundamental systems having fixed the sign of two fundamental parameters. All sections of this chapter are structured similarly. First we collect the results about singular points (both finite and infinite) and limit cycles. Then we locate the rest of the separatrices of the system and we describe the behaviour of the canonical regions. Finally organize all the information in propositions which describe and classify fundamental systems when we vary the two parameters which have been not fixed. At the end of each section we describe the bifurcations which take place in this class of fundamental systems having fixed the sign of two fundamental parameters. We also provide a picture of the parameter space representing the bifurcation manifolds and the corresponding phase portraits.

Readers who are interested only in the results can go directly first to Chpater 1 and after to Chapter 5 where they will find at the end of each section a complete list with all phase portraits and with their bifurcations.

The contents of the book have been arranged for obtainig the full classification of the global dynamics of the fundamental systems using the qualitative theory of EDOs. The authors understand that for readers who only want to learn how to use the qualitative theory for studying ODEs, do not need to follow completely this arranged. The many cases that much be consider for obtainig the global dynamics of all fundamental systems requires that some propositions are very similar to each other and to follow all of them become a little tedious. In a first reading to eliminate this potential problem, the authors recommend that in the Sections 3.11, 4.4 and 4.5, these readers select some of the results in order to know the qualitative arguments that are used in their proofs and leave the rest of the results of these sections for a more detailed reading. It also is recommended in a first approximation to focus on only one of the classe of fundamental systems having fixed the sign of two fundamental parameters in Chapter 5.

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Barcelona, 2010.