ON THE INTEGRABILITY OF VECTOR FIELDS IN $\mathbb{R}^N$

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ABSTRACT. We give necessary and sufficient conditions for the integrability of first order N–dimensional differential systems.

We propose a new method to determine the last $N-1$ first integral for the completely integrability of an N–dimensional differential system with $N-2$ independent first integrals and with a Jacobi multiplier.

As an application we study the integrability of some 3–dimensional differential systems, more precisely the integrability of the asymmetric May–Leonard differential systems, of the symmetric May–Leonard differential systems, and of the Clebsch vector fields.

1. Introduction

For the N–dimensional nonlinear differential systems the existence of $K < N-1$ independent first integrals means that systems is partially integrable. The existence of $N-1$ independent first integrals means that the system is completely integrable, i.e. the intersection of the $N-1$ hypersurfaces obtained fixing the $N-1$ first integrals provide the trajectories of the differential system.

The following natural question arises: *Given a differential system of ordinary differential equations depending on parameters, how does one recognize the values of the parameters for which the system has a first integral*?

In order to give a partial answer to this question we present the following results.

First we give necessary and sufficient conditions under which the differential system

$$\dot{x}_j = X_j(x_1, \ldots, x_N), \quad \text{for} \quad j = 1, \ldots, N, \tag{1}$$

or its associated vector field

$$\mathcal{X} = X_1 \frac{\partial}{\partial x_1} + X_2 \frac{\partial}{\partial x_2} + \ldots + X_N \frac{\partial}{\partial x_N},$$

is completely integrable. Here $X_j : U \longrightarrow \mathbb{R}^N$ are C^1 functions defined in an open subset $U \subseteq \mathbb{R}^N$. Using these necessary and sufficient conditions we propose a new method to determine the last $N-1$ first integral for the integrability of the differential system (1) having $N-2$ independent first integrals and a Jacobi multiplier.

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