RATIONAL FIRST INTEGRALS OF THE LIÉNARD EQUATIONS: THE SOLUTION TO THE POINCARÉ PROBLEM FOR THE LIÉNARD EQUATIONS

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ABSTRACT. Poincaré in 1891 asked about the necessary and sufficient conditions in order to characterize when a polynomial differential system in the plane has a rational first integral. Here we solve this question for the class of Liénard differential equations $\ddot{x} + f(x)\dot{x} + x = 0$, being $f(x)$ a polynomial of arbitrary degree. As far as we know it is the first time that all rational first integrals of a relevant class of polynomial differential equations of arbitrary degree has been classified.

1. THE POINCARÉ PROBLEM ON THE RATIONAL FIRST INTEGRALS OF THE POLYNOMIAL DIFFERENTIAL SYSTEMS

A *rational function* $f(x, y)/g(x, y)$ has degree m if the polynomials $f(x, y)$ and $g(x, y)$ are coprime in the ring $\mathbb{R}[x, y]$, and the maximum of the degrees of $f(x, y)$ and $g(x, y)$ is m.

A *polynomial differential system* is a differential system of the form

$$\frac{dx}{dt} = \dot{x} = P(x, y), \qquad \frac{dy}{dt} = \dot{x} = Q(x, y), \tag{1}$$

where $P(x, y)$ and $Q(x, y)$ are real polynomials in the variables x and y, and t is the independent variable usually called the *time*. The *polynomial vector field* associated to the polynomial differential system (1) is

$$\mathcal{X} = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}.$$

Let U be an open subset of $\mathbb{R}^2$. Here a *first integral* is a $\mathcal{C}^1$ non–locally constant function $H : U \to \mathbb{R}$ such that it is constant on the solutions $(x(t), y(t))$ of the polynomial differential system (1) contained in U, i.e. if $\mathcal{X}(H)|_U \equiv 0$.

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