PHASE PORTRAITS OF BERNOULLI QUADRATIC POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract. In this paper we study a new class of quadratic polynomial differential systems. We classify all global phase portraits in the Poincaré disk of Bernoulli quadratic polynomial differential systems in $\mathbb{R}^2$.

1. Introduction

Quadratic polynomial differential systems appear frequently in many areas of applied mathematics, electrical circuits, astrophysics, in population dynamics, chemistry, neural networks, laser physics, hydrodynamics, etc. Although these differential systems are the simplest nonlinear polynomial systems, they are also important as a basic testing ground for the general theory of the nonlinear differential systems.

There are more than one thousand papers written on the quadratic polynomial differential systems. For example there is a bibliography of some of these compiled by Reyn which has 426 items plus 55 preprints and 10 Reports published in TUDelft series of reports in 1989. See the books of Ye Yanqian et al. [24], Reyn [20], and Artés, Llibre, Schlomiuk and Vulpe [2] dedicated to the quadratic polynomial differential systems. See also the classical surveys on these systems by Coppel [6], and Chicone and Jinghuang [5].

Consider the differential equation

$$\frac{dy}{dx} = A(x)y^k + B(x)y, \tag{1.1}$$

with $k \in \mathbb{R} \setminus \{0,1\}$ and A, B non zero real functions. This differential equation is called Bernoulli differential equation. Associated to the Bernoulli differential equation we can define the Bernoulli differential system given by

$$\begin{aligned} \dot{x} &= p(x), \\ \dot{y} &= a(x)y^k + b(x)y. \end{aligned} \tag{1.2}$$

Note that system (1.2) is equivalently equation (1.1).

In this paper we consider Bernoulli polynomial differential system of degree 2 in $\mathbb{R}^2$, i.e. $p(x)$ is a polynomial with degree at most 2, $k = 2$, $a(x)$ is a constant non zero, and $b(x)$ is a non zero polynomial of degree at most 1 (otherwise the

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system (1.2) will be of separable variables). Thus our objective is to classify all phase portraits of the system

$$\begin{aligned} \dot{x} &= ax^2 + bx + c, \\ \dot{y} &= dy^2 + (ex + f)y, \end{aligned} \tag{1.3}$$

with $d(e^2 + f^2) \neq 0$.

The topological phase portraits in the Poincaré of many classes of quadratic polynomial differential systems have classified. One of the first classes analyzed was the classification of the quadratic centers which started with the works of Dulac [8], Kapteyn [11, 12], Bautin [4], Schlomiuk [21], Żołądek [26], Ye and Ye [25], Artés, Llibre and Vulpe [3], ... The class of the homogeneous quadratic systems by Lyagina [14], Markus [15], Korol [13], Sibirskii and Vulpe [22], Newton [17], Date [7] and Vdovina [23],... The class of Hamiltonian quadratic systems, see Artés and Llibre [1], Kalin and Vulpe [10] and Artés, Llibre and Vulpe [3], etc.

Our main result is the following one.

Theorem 1.1. *The phase portraits in the Poincaré disk of system* (1.3) *are topologically equivalent to one of the* 22 *phase portraits presented in Figures* 1 (*except Figure* 1 (*d*)), 2 (*except Figure* 2 (*b*)), 3 *and* 4.

The proof of above theorem is given in the end of Section 6.

2. Definitions and useful results

Let U an open subset of $\mathbb{R}^2$ and $X : U \to \mathbb{R}^2$ a vector field. If $(x_0, y_0) \in U$ is a singular point of X, we say that (x_0, y_0) is a *hyperbolic singular point* when the real part of both eigenvalues of $DX(x_0, y_0)$ are different of zero. If $DX(x_0, y_0)$ has exactly one of the eigenvalues different of zero, we say that (x_0, y_0) is *semi-hyperbolic singular point* of X. The point (x_0, y_0) is called a *elementary singular point of* X if (x_0, y_0) is a hyperbolic or a semi-hyperbolic singular point of X, otherwise (x_0, y_0) is called a *non-elementary singular point of* X.

In this work to classify topologically the singular points of X, we use the definitions of node and saddle points (with their stability), also elliptic, hyperbolic and parabolic sectors (attracting or repelling) as in [19]. For analyzing the topological behavior of the flow near a hyperbolic singular point of X, we use the classical theory of dynamical systems and if we want to analyze the behavior of the flow near a semi-hyperbolic singular point we use Theorem 1 of page 151 from [19].

Now we say that a non-elementary singular point (x_0, y_0) is a *nilpotent singularity* of X if $DX(x_0, y_0)$ has both the eigenvalues equals to zero, but $DX(x_0, y_0)$ is not zero. Information on this nilpotent singular points can be find in Theorem 3.5 of [9]. Now, if $DX(x_0, y_0)$ is the null matrix then (x_0, y_0) is a *linearly zero singularity.*

To study the local phase portraits of the linearly zero singular points, we do *blow-ups* consisting of a change of coordinates of the form $x \mapsto x, y \mapsto xy$, and $x \mapsto xy, y \mapsto y$ (for more details, see page 91 of [9]).

3. Poincaré Compactification

In the study of trajectories of polynomial vector fields, is essential to understand the behavior of solutions escaping to infinity and a important tool for this is the compactification technique. In short, this method consists of extend analytically the vector field to a compact manifold, in fact to a sphere. We identify $\mathbb{R}^n$ with northern and southern hemispheres through simple projections, then the vector field X in $\mathbb{R}^n$ can be extended to a vector field $\overline{X}$ in $\mathbb{S}^n$. This method is called the *Poincaré compactification*. We describe below this method when $n = 2$, more details, see [9].

Let X be the polynomial vector field defined on $\mathbb{R}^2$ by system

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y),$$

where P and Q are polynomials in the variables x and y with real coefficients. The the *degree* of the polynomial vector field X is defined by $d = \max\{\deg P, \deg Q\}$.

We denote by $\mathbb{S}^2 = \{(z_1, z_2, z_3) \in \mathbb{R}^3; z_1^2 + z_2^2 + z_3^2 = 1\}$ and $\mathbb{S}^1 = \{(z_1, z_2, z_3) \in \mathbb{S}^2; z_3 = 0\}$. We identify $\mathbb{R}^2$ as the plane $z_3 = 1$, i.e., the tangent plane π of $\mathbb{S}^2$ at the north pole $(0, 0, 1)$, and using the central projection of π in $\mathbb{S}^2$, we obtain a tangent vector field defined on $\mathbb{S}^2 \backslash \mathbb{S}^1$ such that the infinity points of π are projected in $\mathbb{S}^1$.

In general, this vector field is unbounded near $\mathbb{S}^1$ and symmetric about the center of $\mathbb{S}^2$. But this vector field admits an unique analytical extension to $\mathbb{S}^2$, after of a multiplication by an appropriate factor. This analytical extension is called *the Poincaré compactification of* X and denoted by $p(X)$. For study $p(X)$, due the symmetry, is sufficient to consider its restriction to the closed northern hemisphere H of $\mathbb{S}^2$. We call the *Poincaré disk* the orthogonal projection of H into the disk $\{(z_1, z_2, z_3) \in \mathbb{R}^3; z_1^2 + z_2^2 \leq 1, z_3 = 0\}$.

In each hemisphere we have that $p(X)$ is C^ω-equivalent, but not C^ω-conjugated, to X. Then the singular points of X correspondent singularities of $p(X)$, but may be that $p(X)$ has singularities in $\mathbb{S}^1$. A singular point of $p(X)$ which belongs to $\mathbb{S}^2 \setminus \mathbb{S}^1$ (respectively $\mathbb{S}^1$) is called *finite (respectively infinite) singular point of* X. Moreover, we have that $\mathbb{S}^1$ is invariant under the flow of $p(X)$.

To obtain expressions of $p(X)$ in local coordinates, we consider the charts of the sphere $\mathbb{S}^2$. For $j = 1, 2, 3$ define $U_j = \{(z_1, z_2, z_3) \in \mathbb{S}^2; z_j > 0\}$, $V_j = \{(z_1, z_2, z_3) \in \mathbb{S}^2; z_j < 0\}$ and $\varphi_j : U_j \to \mathbb{R}^2, \psi_j : V_j \to \mathbb{R}^2$ given by

$$\varphi_1(z) = -\psi_1(z) = \frac{(z_2, z_3)}{z_1}, \;\; \varphi_2(z) = -\psi_2(z) = \frac{(z_1, z_3)}{z_2}, \;\; \varphi_3(z) = \frac{(z_1, z_2)}{z_3}.$$

If we denote by (u, v) the value of φ_j or ψ_j at the point z we can prove that the expression of $p(X)$ in the chart (U_1, φ_1) is given by

$$\dot{u} = v^d \left[-uP\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right) \right], \quad \dot{v} = -v^{d+1} P\left(\frac{1}{v}, \frac{u}{v}\right).$$

The expression of $p(X)$ in the chart (U_2, φ_2) is

$$\dot{u} = v^d \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right) \right], \quad \dot{v} = -v^{d+1} Q\left(\frac{u}{v}, \frac{1}{v}\right),$$

and the expression of $p(X)$ in the chart (U_3, φ_2) is

$$\dot{u} = P(u, v), \quad \dot{v} = Q(u, v).$$

Finally, for each $j = 1, 2, 3$, the expression of $p(X)$ in the chart (V_j, ψ_j) is the expression of $p(X)$ in the chart (U_j, φ_j) multiplied by the factor $(-1)^{d-1}$.

Using this notation we observe that if $(u, v) \in U_j$ is an infinite singular point of X if, and only if, the expression of $p(X)$ in the chart (U_j, φ_j) vanishes in (u, v) and $v = 0$.

Observe that if z is an infinite singular point of X then $-z$ is also an infinite singular point of X. In this case, from the expressions of $p(X)$ in local coordinates it follows that the behavior of the flow near $-z$ can be determined by the behavior of the flow near z, because the flow near $-z$ differs by the flow near z by the factor $(-1)^{d-1}$. Then the study of $p(X)$ in the charts $(V_j, \psi_j), j = 1, 2, 3$, is superfluous. Moreover, notice that if z is an infinite singular point of X with $z \in U_2, z \neq (0, 1, 0)$ then $z \in U_1 \cup V_1$. It follows that to study all the infinite singular points of X, it is sufficient to study the singularities of $p(X)$ in U_1 and the origin of U_2.

4. Markus-Neumann-Peixoto Theorem

The study of the phase portrait of a given planar vector fields can be reduced to the determination of the separatrices (see definition below) and a finite number of special orbits. This result is known as Markus-Neumann-Peixoto Theorem, for more details see [15], [16], [19] or p. 33 of [9].

Let X an Y be C^1-vector fields defined on the open sets U and V of $\mathbb{R}^2$, respectively. Denote by (U, Φ) and (V, Ψ) the flow of X and Y, respectively. We say that (U, Φ) and (V, Ψ) are *topologically equivalent* if there exists a homeomorphism of U in V which carries the orbits of X in orbits of Y, preserving the orientation of the all orbits, and in this case we also say that their phase portraits are *topologically equivalent.*

Consider the following vector fields

- $V = \mathbb{R}^2$ and $Y(x, y) = (1, 0), \forall (x, y) \in \mathbb{R}^2$,
- $V = \mathbb{R}^2 \setminus \{(0, 0)\}$ and Y such that, in polar coordinates, is given by $\dot{r} = 0, \dot{\theta} = 1$,
- $V = \mathbb{R}^2 \setminus \{(0, 0)\}$ and Y such that, in polar coordinates, Y is given by $\dot{r} = r, \dot{\theta} = 0$.

We call the flow of the three vector fields above of *strip flow, annulus flow and nodal flow,* respectively. Now, suppose that $U = \mathbb{R}^2$, if the flow $(\mathbb{R}^2, \Phi)$ is topologically equivalent either to a strip flow or annulus flow or a nodal flow it is called *parallel.*

Denote by $\gamma(p)$ the orbit of $p \in U$, and by $\alpha(p)$ and $\omega(p)$, the respective α-limit and the ω-limit of p. The orbit $\gamma(p)$ is a *separatrix* if

- $\gamma(p)$ is a singular point, or
- $\gamma(p)$ is a periodic orbit and there is no neighborhood of $\gamma(p)$ consisting of periodic orbits, or
- $\gamma(p)$ is homeomorphic to $\mathbb{R}$ and there is no neighborhood W of $\gamma(p)$ with the following two properties:
  - $q \in W \Rightarrow \alpha(q) = \alpha(p)$ and $\omega(q) = \omega(p)$,
  - the boundary of W is composed by $\alpha(p), \omega(p)$ and by two another orbits $\gamma(p_1), \gamma(p_2)$ such that $\alpha(p_1) = \alpha(p_2) = \alpha(p)$ and $\omega(p_1) = \omega(p_2) = \omega(p)$.

We denote by Σ the union of all separatrices of a given flow (U, Φ) , Σ. is called *extended separatrix skeleton,*. Note that it is a closed invariant subset of U and each connected component of $U \setminus \Sigma$ is an open invariant set, called a *canonical region.* There exist only three possibilities for the flow in each canonical region, more precisely we have the following result.

Proposition 4.1. *In each canonical region the flow is parallel.*

The union of the extended separatrix skeleton with one orbit in each canonical region is called *completed separatrix skeleton*. Consider the extended separatrix skeleton C_1 and C_2 of the flows $(\mathbb{R}^2, \Phi)$ and $(\mathbb{R}^2, \Psi)$, respectively. Then, if there exist a homeomorphism of $\mathbb{R}^2$ in $\mathbb{R}^2$ which map orbits of C_1 into orbits of C_2 preserving the orientation, we say that C_1 and C_2 are *topologically equivalent*.

Now we can to present the Markus-Newmann-Peixoto theorem which implies that, to draw the phase portrait of a given planar vector field, it is sufficient determine its completed separatrix skeleton.

Theorem 4.2 (Markus-Neumann-Peixoto)**.** *Consider the continuous flows $(\mathbb{R}^2, \Phi)$ and $(\mathbb{R}^2, \Psi)$ and suppose that they have only isolated singular points. Then $(\mathbb{R}^2, \Phi)$ and $(\mathbb{R}^2, \Psi)$ are topologically equivalent if, and only if, its completed separatrices skeleton are topologically equivalent.*

5. Local Phase Portrait of Finite and Infinite Singular Points

In this section we determinate the local local phase portrait of the finite and infinite singular points of system (1.3).

As in section 3 we denote by $p(X)$ the Poincaré compactification of system (1.3). Here the singular points of $p(X)$ in $\mathbb{S}^1$ will be denoted by q_i. Remember that, if q_i is a singular point of $p(X)$, then $-q_i$ is also. Moreover, as the degree of system (1.3) is two, the behavior of the flow near $-q_i$ is the same of near q_i but reversing the sense of the orbits. Thus we will describe the local phase portrait of the infinite singular points q_i.

In terms of the number, multiplicity and type of the roots of the polynomial $p(x) = ax^2 + bx + c$ of system (1.3), we distinguish five cases.

5.1. Case 1: $p(x)$ has two distinct reals roots.

In this case, we can write system (1.3) as

$$\dot{x} = (x-\alpha)(x-\beta), \qquad \dot{y} = dy^2 + (ex+f)y, \tag{5.1}$$

with $d(e^2+f^2) \neq 0$ and $\alpha \neq \beta$. The singular points of system (5.1) are:

$$p_1 = (\alpha, 0); \quad p_2 = (\beta, 0); \quad p_3 = \left(\alpha, -\frac{e\alpha + f}{d}\right) \text{ and } p_4 = \left(\beta, -\frac{e\beta + f}{d}\right).$$

Denote by λ_i, μ_i, $i = 1, \dots, 4$, the eigenvalues of the linear parts of system (5.1) at the singular point p_i.

The next three results determine the local phase portrait of the finite singular points.

Proposition 5.1. *Suppose that system* (5.1) *has four singular points, i.e., $(e\alpha + f)(e\beta + f) \neq 0$.*

(a) *If $e\alpha + f > 0$, $\alpha - \beta > 0$ and $e\beta + f > 0$, then p_1 is an unstable node, p_3 and p_2 are saddles, and p_4 is a stable node;*
(b) *If $e\alpha + f > 0$, $\alpha - \beta > 0$ and $e\beta + f < 0$, then p_1 is an unstable node, p_3 and p_4 are saddles, and p_2 is a stable node;*
(c) *If $e\alpha + f < 0$, $\alpha - \beta > 0$ and $e\beta + f > 0$, then p_3 is an unstable node, p_1 and p_2 are saddles, and p_4 is a stable node;*
(d) *If $e\alpha + f < 0$, $\alpha - \beta > 0$ and $e\beta + f < 0$, then p_3 is an unstable node, p_1 and p_4 are saddles, and p_2 is a stable node;*

(e) *If* $e\alpha + f > 0$*,* $\alpha - \beta < 0$ *and* $e\beta + f > 0$*, then* p_3 *is a stable node,* p_1 *and* p_4 *are saddles, and* p_2 *is an unstable node;*
(f) *If* $e\alpha + f > 0$*,* $\alpha - \beta < 0$ *and* $e\beta + f < 0$*, then* p_3 *is a stable node,* p_1 *and* p_2 *are saddles, and* p_4 *is an unstable node;*
(g) *If* $e\alpha + f < 0$*,* $\alpha - \beta < 0$ *and* $e\beta + f > 0$*, then* p_1 *is a stable node,* p_3 *and* p_4 *are saddles, and* p_2 *is an unstable node;*
(h) *If* $e\alpha + f < 0$*,* $\alpha - \beta < 0$ *and* $e\beta + f < 0$*, then* p_1 *is a stable node,* p_3 *and* p_2 *are saddles, and* p_4 *is an unstable node.*

Proof. We have that $\lambda_1 = e\alpha+f$, $\mu_1 = \alpha-\beta$, $\lambda_2 = e\beta+f$, $\mu_2 = \beta-\alpha$, $\lambda_3 = -e\alpha-f$, $\mu_3 = \alpha - \beta$, $\lambda_4 = -e\beta - f$ and $\mu_4 = \beta - \alpha$. Therefore, the rest of the proof follows of the fact that all the singular points are hyperbolic, and then its local phase portraits are known. □

Proposition 5.2. *Suppose that system* (5.1) *has exactly three singular points, i.e.,* $(e\alpha + f)(e\beta + f) = 0$ *and* $(e\alpha + f)^2 + (e\beta + f)^2 \neq 0$*.*
(a) *If* $e\alpha + f = 0$*,* $\alpha - \beta > 0$ *and* $e\beta + f > 0$*, then* p_1 *is a saddle-node,* p_2 *is a saddle, and* p_4 *is a stable node;*
(b) *If* $e\alpha + f = 0$*,* $\alpha - \beta > 0$ *and* $e\beta + f < 0$*, then* p_1 *is a saddle-node,* p_2 *is a stable node, and* p_4 *is a saddle;*
(c) *If* $e\alpha + f = 0$*,* $\alpha - \beta < 0$ *and* $e\beta + f > 0$*, then* p_1 *is a saddle-node,* p_2 *is an unstable node, and* p_4 *is a saddle;*
(d) *If* $e\alpha + f = 0$*,* $\alpha - \beta < 0$ *and* $e\beta + f < 0$*, then* p_1 *is a saddle-node,* p_2 *is a saddle, and* p_4 *is an unstable node;*
(e) *If* $e\alpha + f > 0$*,* $\alpha - \beta > 0$ *and* $e\beta + f = 0$*, then* p_1 *is an unstable node,* p_2 *is a saddle-node, and* p_3 *is a saddle;*
(f) *If* $e\alpha + f < 0$*,* $\alpha - \beta > 0$ *and* $e\beta + f = 0$*, then* p_1 *is a saddle,* p_2 *is a saddle-node, and* p_3 *is an unstable node;*
(g) *If* $e\alpha + f > 0$*,* $\alpha - \beta < 0$ *and* $e\beta + f = 0$*, then* p_1 *is a saddle,* p_2 *is a saddle-node, and* p_3 *is a stable node;*
(h) *If* $e\alpha + f < 0$*,* $\alpha - \beta < 0$ *and* $e\beta + f = 0$*, then* p_1 *is a stable node,* p_2 *is a saddle-node, and* p_3 *is a saddle.*

Proof. First we suppose that $e\alpha + f = 0$, so the eigenvalues associated to singular points $p_1 = (\alpha, 0)$ are $\lambda_1 = 0$ and $\mu_1 = \alpha - \beta$. Now, doing the change of coordinates $(x, y, t) \mapsto \left(u + \alpha, v, \dfrac{s}{\alpha - \beta}\right)$, system (5.1) becomes

$$\begin{aligned} u' &= u + \frac{1}{\alpha - \beta}u^2 = u + P(u, v), \\ v' &= \frac{e}{\alpha - \beta}uv + \frac{d}{\alpha - \beta}v^2 = Q(u, v), \end{aligned}$$

and so p_1 correspond to origin. Note that $u \equiv 0$ is the solution of equation $u + P(u, v) = 0$ and $Q(0, v) = \dfrac{d}{\alpha - \beta}v^2$. Hence, by Theorem 1 from the page 151 of [19], we have that p_1 is a saddle-node. For the others two singularities p_2 and p_4, it follows that $\lambda_2 = e\beta + f$, $\mu_2 = \beta - \alpha$, $\lambda_4 = -e\beta - f$ and $\mu_4 = \beta - \alpha$. Therefore, the rest of the proof of statements (a), (b), (c) and (d) follows taking into account the signs of the eigenvalues because these points are hyperbolic.

The proof of case $e\beta + f = 0$, i.e., statements (e), (f), (g) and (h) is analogous to the previous case. □

Proposition 5.3. *Suppose that system* (5.1) *has exactly two singular points, i.e.,* $e\alpha + f = 0$ *and* $e\beta + f = 0$. *Then the singular points are saddle-nodes.*

Proof. The eigenvalues associated to singularities $p_1 = (\alpha, 0)$ and $p_2 = (\beta, 0)$ are $\lambda_1 = 0$, $\mu_1 = \alpha - \beta$, $\lambda_2 = 0$ and $\mu_2 = \beta - \alpha$, respectively. In this case, since $\alpha \neq \beta$, we have $e = 0$. Now, doing the change of coordinates $(x, y, t) \mapsto \left(u + \alpha, v, \dfrac{s}{\alpha - \beta}\right)$, system (5.1) if $e = 0$, becomes

$$u' = u + \frac{1}{\alpha - \beta}u^2 = u + P(u, v), \quad v' = \frac{d}{\alpha - \beta}v^2 = Q(u, v),$$

and so p_1 corresponds to the origin. Note that $u \equiv 0$ is the solution of equation $u + P(u, v) = 0$ and $Q(0, v) = \dfrac{d}{\alpha - \beta}v^2$. Hence, by Theorem 1 of page 151 of [19], we have that p_1 is a saddle-node. Analogously we have p_2 is a saddle-node. □

The next result determine the local phase portrait of the infinite singular points.

Proposition 5.4. *Let* $p(X)$ *be the Poincaré compactification of system* (5.1).

(a) *If* $1 - e \neq 0$, *then* $p(X)$ *has six singularities* $\pm q_1$, $\pm q_2$ *and* $\pm q_3$ *in the equator* $\mathbb{S}^1$. *Moreover,* q_1 *is a saddle (resp. stable node) and* q_2 *is a stable node (resp. saddle) if* $1 - e < 0$ *(resp.* $1 - e > 0$*), and* q_3 *is either a stable node when* $d > 0$, *or an unstable node when* $d < 0$.

(b) *If* $1 - e = 0$, *then* $p(X)$ *has four singularities* $\pm q_1$ *and* $\pm q_3$ *in the equator* $\mathbb{S}^1$. *Moreover* q_1 *is a saddle-node and* q_3 *is either a stable node when* $d > 0$ *or an unstable node when* $d < 0$.

Proof. The system associated to $p(X)$ in the charts U_1 and U_2 are

$$\begin{aligned} u' &= (-1 + e)\,u + du^2 + (\beta + \alpha + f)\,uv - \alpha\beta uv^2, \\ v' &= -v + (\alpha + \beta)v^2 - \alpha\beta v^3, \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} u' &= -du + (1 - e)\,u^2 - (\beta + \alpha + f)\,uv + \alpha\beta v^2, \\ v' &= -dv - euv - fv^3, \end{aligned} \tag{5.3}$$

respectively.

In the chart U_1 for $v = 0$ we have the singular points $q_1 = (0, 0)$ and $q_2 = \left(\dfrac{1 - e}{d}, 0\right)$ of system (5.2). The eigenvalues associated to q_1 and q_2 are $\lambda_{11} = e - 1$, $\lambda_{12} = -1$ and $\lambda_{21} = 1 - e$, $\lambda_{22} = -1$, respectively. Now in the chart U_2, $q_3 = (0, 0)$ is a singular points of system (5.3), and its eigenvalues are $\lambda_{31} = \lambda_{32} = -d$. Therefore, the proof of the statement (a) follows by studying the signs of the eigenvalues.

For case $1 - e = 0$, system (5.2) becomes, after a time rescaling,

$$\begin{aligned} u' &= -du^2 - (\beta + \alpha + f)\,uv + \alpha\beta uv^2 = P(u, v), \\ v' &= v - (\alpha + \beta)v^2 + \alpha\beta v^3 = v + Q(u, v). \end{aligned} \tag{5.4}$$

Note that in this case $q_1 = q_2$ and, in the chart U_1, q_1 correspond to the singular point at the origin of system (5.4) with eigenvalues $\lambda_{11} = 0$ and $\lambda_{12} = 1$. As $v \equiv 0$ is the solution of equation $v + Q(u, v) = 0$ and $P(u, 0) = -du^2$. By Theorem 1 of page 151 of [19], we have that q_1 is a saddle-node. Hence statement (b) follows. □

5.2. **Case 2: $p(x)$ has a double real root.** In this case we can write system (1.3) as

$$\dot{x} = (x-\alpha)^2, \qquad \dot{y} = dy^2 + (ex+f)y. \tag{5.5}$$

The singular points of system (5.5) are:

$$p_1 = (\alpha, 0) \text{ and } p_3 = \left(\alpha, -\frac{e\alpha + f}{d}\right). \tag{5.6}$$

Denote by λ_i, μ_i, $i = 1, 3$, the eigenvalues of the linear parts of system (5.5) at the singular point p_i.

The next result determines the local phase portrait of the finite singular points.

Proposition 5.5. *Consider system* (5.5).

(a) *If $e\alpha + f \neq 0$, then the singular points p_1 and p_3 are distinct and both are saddle-nodes.*
(b) *If $e\alpha + f = 0$, then $p_1 = p_3$ and it is a singular point with two parabolic sectors and two hyperbolic sectors.*

Proof. First we suppose that $e\alpha + f \neq 0$, then $\lambda_1 = 0$, $\mu_1 = e\alpha + f$, $\lambda_3 = 0$ and $\mu_3 = -(e\alpha + f)$. Doing the change of variables $(x, y, t) \mapsto \left(u + \alpha, v, \dfrac{s}{f + \alpha e}\right)$, system (5.5) becomes

$$\begin{aligned} u' &= \frac{u^2}{f + \alpha e} = P(u, v), \\ v' &= v + \frac{e}{f + \alpha e} uv + \frac{d}{f + \alpha e} v^2 = v + Q(u, v), \end{aligned}$$

and so p_1 corresponds to the origin. Note that $v \equiv 0$ is the solution of equation $v + Q(u, v) = 0$ and $P(u, 0) = \dfrac{1}{f + \alpha e} u^2$. Hence, by Theorem 1 of page 151 of [19], we have that p_1 is a saddle-node.

Now doing the change of variables $(x, y, t) \mapsto \left(-dv + \alpha, u + ev - \dfrac{f + \alpha e}{d}, -\dfrac{s}{f + \alpha e}\right)$, system (5.5) becomes

$$\begin{aligned} u' &= u - \frac{d}{f + \alpha e} u^2 - \frac{ed}{f + \alpha e} uv - \frac{ed}{f + \alpha e} v^2 = u + P(u, v), \\ v' &= \frac{d}{f + \alpha e} v^2 = Q(u, v), \end{aligned}$$

and so p_3 corresponds to origin. Analogously to the previous case, we have that p_3 is a saddle-node.

When $e\alpha + f = 0$, by (5.6) we have $p_1 = p_3$ and $\lambda_1 = \mu_1 = 0$. Hence, doing the change of variables $(x, y) \mapsto (u + \alpha, v)$, system (5.5) with $f = -\alpha e$ becomes

$$u' = u^2, \qquad v' = euv + dv^2. \tag{5.7}$$

As $(0, 0)$ is a linearly zero singular point of system (5.7), we will doing a blow-up in the direction u. More precisely, doing $u = \tilde{x}$ and $v = \tilde{x}\tilde{y}$ in system (5.7) and after a time rescaling, we obtain

$$\tilde{x}' = \tilde{x}, \qquad \tilde{y}' = (e-1)\tilde{y} + d\tilde{y}^2. \tag{5.8}$$

When $e-1 \neq 0$ system (5.8) has two singularities $\tilde{p}_1 = (0,0)$ and $\tilde{p}_3 = \left(0, \dfrac{1-e}{d}\right)$ with respective eigenvalues $\tilde{\lambda}_1 = 1$, $\tilde{\mu}_1 = e-1$, $\tilde{\lambda}_3 = 1$ and $\tilde{\mu}_3 = 1-e$. If $e-1 > 0$ (resp. $e-1 < 0$), then $\tilde{p}_1$ is an unstable node (resp. saddle), and $\tilde{p}_3$ is a saddle (resp. unstable node).

For $e-1=0$, $\tilde{p}_1$ is the unique singularity of system (5.8), and by Theorem 1 of page 151 of [19], we have that $\tilde{p}_1$ is a saddle-node.

Now we do a blow-up in the direction v. More precisely, doing $u = \tilde{x}\tilde{y}$ and $v = \tilde{y}$ in system (5.7) and after a time rescaling, we obtain

$$\tilde{x}' = -d\tilde{x} + (1-e)\tilde{x}^2, \qquad \tilde{y}' = d\tilde{y} + e\tilde{x}\tilde{y}. \tag{5.9}$$

We have to study only the singular point $\tilde{q}_3 = (0,0)$ of system (5.9). This singular point has eigenvalues $\pm d$, and so $\tilde{q}_3$ is a saddle. In summary, going back through the blow ups, p_1 is a singular point with two hyperbolic sectors and two parabolic sectors. □

The local phase portraits of the infinite singular points in this case are the same obtained in Case 1. In fact, the Poincaré compactification of system (5.5) in the charts U_1 and U_2 are given by systems (5.2) and (5.3) doing $\alpha = \beta$, respectively. Therefore, we have the same result as Proposition 5.4 whose the proof is analogous. The result is the following.

Proposition 5.6. *Let $p(X)$ be the Poincaré compactification of system* (5.5)*.*

(a) *If $1-e \neq 0$, then $p(X)$ has six singularities $\pm q_1$, $\pm q_2$ and $\pm q_3$ in the equator $\mathbb{S}^1$. Moreover, q_1 is a saddle (resp. stable node) and q_2 is a stable node (resp. saddle) if $1-e<0$ (resp. $1-e>0$), and q_3 is either a stable node when $d>0$, or an unstable node when $d<0$.*

(b) *If $1-e=0$, then $p(X)$ has four singularities $\pm q_1$ and $\pm q_3$ in the equator $\mathbb{S}^1$. Moreover q_1 is a saddle-node and q_3 is either a stable node when $d>0$ or an unstable node when $d<0$.*

5.3. **Case 3: $p(x)$ has only one real root.** In this case, we can write system (1.3) as

$$\dot{x} = x - \alpha, \qquad \dot{y} = dy^2 + (ex+f)y. \tag{5.10}$$

The singular points of system (5.10) are:

$$p_1 = (\alpha, 0) \text{ and } p_2 = \left(\alpha, -\frac{e\alpha+f}{d}\right). \tag{5.11}$$

Denote by λ_i, μ_i, $i=1,2$, the eigenvalues of the linear parts of system (5.10) at the singular point p_i.

The next result determines the local phase portrait of the finite singular points.

Proposition 5.7. *Consider system* (5.10)*.*

(a) *If $e\alpha+f>0$ (resp. $e\alpha+f<0$), then the singular point p_1 is an unstable node (resp. saddle) and p_2 is saddle (resp. unstable node).*

(b) *If $e\alpha+f=0$, then $p_1=p_2$ and it is a saddle-node.*

Proof. When $e\alpha+f \neq 0$, we have $\lambda_1 = 1$, $\mu_1 = e\alpha+f$, $\lambda_2 = 1$, $\mu_2 = -(e\alpha+f)$. Therefore the proof of statement (a) follows from the signs of the eigenvalues.

Now if $e\alpha + f = 0$, by (5.11) we have $p_1 = p_2$ and $\lambda_1 = 1$ and $\mu_1 = 0$. Hence doing the change of variables $(x, y) \mapsto (u + \alpha, v)$, system (5.10) with $f = -\alpha e$ becomes

$$u' = u, \qquad v' = euv + dv^2.$$

Hence by Theorem 1 of page 151 of [19], we have that $\tilde{p}_1$ is a saddle-node. Statement (b) is proved. □

The next result determines the local phase portrait of the infinite singular points.

Proposition 5.8. *Let $p(X)$ be the Poincaré compactification of system* (5.10).

(a) *If $e \neq 0$, then $p(X)$ has six singularities $\pm q_1$, $\pm q_2$ and $\pm q_3$ in the equator $\mathbb{S}^1$. Moreover, $\pm q_1$, $\pm q_2$ are saddle-nodes and q_3 is stable (resp. unstable) node when $d > 0$ (resp. $d < 0$).*

(b) *If $e = 0$, then $p(X)$ has four singularities $\pm q_1$ and $\pm q_3$ in the equator $\mathbb{S}^1$. Moreover q_1 is a singular point with two hyperbolic sectors and two parabolic sectors, and q_3 is either a stable node when $d > 0$, or an unstable node when $d < 0$.*

Proof. The system associated to $p(X)$ in the charts U_1 and U_2 are

$$\begin{aligned} u' &= eu + du^2 + (f-1)uv + \alpha uv^2, \\ v' &= -v^2 + \alpha v^3, \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} u' &= -du - eu^2 + (1-f)uv - \alpha v^2, \\ v' &= -dv - euv - fv^2, \end{aligned} \tag{5.13}$$

respectively.

In the chart U_2, $q_3 = (0, 0)$ is a singular point of system (5.13), and its eigenvalues are $\lambda_{31} = \lambda_{32} = -d$. Therefore q_3 is stable (resp. unstable) node when $d > 0$ (resp. $d < 0$).

We suppose $e \neq 0$. In the chart U_1 for $v = 0$ we have the singular points $q_1 = (0, 0)$ and $q_2 = \left(-\frac{e}{d}, 0\right)$ of system (5.12). The eigenvalues associated to q_1 and q_2 are $\lambda_{11} = e$, $\lambda_{12} = 0$ and $\lambda_{21} = -e$, $\lambda_{22} = 0$, respectively. By Theorem 1 of page 151 of [19], we have that q_1 is a saddle-node. Analogously, after the change of variables $(u, v, t) \mapsto \left(x + \frac{1-f}{d}y - \frac{e}{d}, y, -\frac{s}{e}\right)$ applied to system (5.12), we obtain that q_2 is a saddle-node. This proves statement (a).

For the case $e = 0$ in the chart U_1 we have that $q_1 = q_2 = (0, 0)$ is a linearly zero singular point. We do a blow-up in the direction u. More precisely, doing $u = \tilde{x}$ and $v = \tilde{x}\tilde{y}$ in system (5.12) and after a time rescaling, we obtain

$$\tilde{x}' = d\tilde{x} + (f-1)\tilde{x}\tilde{y} + \alpha\tilde{x}^2\tilde{y}^2, \qquad \tilde{y}' = -d\tilde{y} - f\tilde{y}^2. \tag{5.14}$$

When $f \neq 0$, system (5.14) has two singularities $\tilde{q}_1 = (0, 0)$ and $\tilde{q}_2 = \left(0, -\frac{d}{f}\right)$ with respective eigenvalues $\tilde{\lambda}_1 = d$, $\tilde{\mu}_1 = -d$, $\tilde{\lambda}_2 = \frac{d}{f}$ and $\tilde{\mu}_2 = d$. Note that $\tilde{q}_1$ is always a saddle. Now $\tilde{q}_2$ is either a saddle if $f < 0$, or an unstable (resp. stable) node if $f > 0$ and $d > 0$ (resp. $f > 0$ and $d < 0$).

Now when $f = 0$, $\tilde{q}_1$ is a unique singularity of system (5.14) ,and as in the previous case it is a saddle.

We do a blow-up in the direction v. More precisely, doing $u = \tilde{x}\tilde{y}$ and $v = \tilde{y}$ in system (5.12) and after a time rescaling, we obtain

$$\tilde{x}' = f\tilde{x} + d\tilde{x}^2, \qquad \tilde{y}' = -\tilde{y} + \alpha\tilde{y}^2. \tag{5.15}$$

We study only the singular point $\tilde{q}_3 = (0,0)$ of system (5.15). This singular point has eigenvalues $\tilde{\lambda}_3 = f$ and $\tilde{\mu}_3 = -1$, and so $\tilde{q}_3$ is either a saddle if $f > 0$, or a stable node if $f < 0$, or (by Theorem 1 of page 151 of [19]) a saddle-node if $f = 0$.

In short, going back through the blow-ups we get that q_1 is a singular point with two hyperbolic sectors and two parabolic sectors. So statement (b) is proved. □

5.4. **Case 4: $p(x)$ is constant.** In this case, we can write system (1.3) as

$$\dot{x} = 1, \qquad \dot{y} = dy^2 + (ex + f)y. \tag{5.16}$$

Note that system (5.16) does not have finite singular points. The next result determines the local phase portrait of the infinite singular points.

Proposition 5.9. *Let be $p(X)$ be in the equatorthe Poincaré compactification of system* (5.16).

(a) *If $e \neq 0$, then $p(X)$ has six singularities $\pm q_1$, $\pm q_2$ and $\pm q_3$ in the equator $\mathbb{S}^1$. Moreover, q_1 is a topological saddle (resp. stable node) if $e > 0$ (resp. $e < 0$), q_2 is a topological saddle (resp. stable node) if $e < 0$ (resp. $e > 0$), and q_3 is stable (resp. unstable) node when $d > 0$ (resp. $d < 0$).*

(b) *If $e = 0$ and $f \neq 0$, then $p(X)$ has four singularities $\pm q_1$ and $\pm q_3$ in the equator $\mathbb{S}^1$. Moreover q_1 is a saddle-node and q_3 is either a stable node when $d > 0$, or an unstable node when $d < 0$.*

(c) *If $e = 0$ and $f = 0$, then $p(X)$ has four singularities $\pm q_1$ and $\pm q_3$ in the equator $\mathbb{S}^1$. Moreover q_1 is a singular point with two hyperbolic sectors and two parabolic sectors and q_3 is either a stable node when $d > 0$, or an unstable node when $d < 0$.*

Proof. The system associated to $p(X)$ in the charts U_1 and U_2 are

$$u' = eu + du^2 + fuv - uv^2, \qquad v' = -v^3, \tag{5.17}$$

and

$$u' = -du - eu^2 - fuv + v^2, \qquad v' = -dv - euv - fv^2, \tag{5.18}$$

respectively.

In the chart U_2, $q_3 = (0,0)$ is a singular points of system (5.18), and its eigenvalues are $\lambda_{31} = \lambda_{32} = -d$. Therefore q_3 is stable (resp. unstable) node when $d > 0$ (resp. $d < 0$).

We suppose $e \neq 0$. In the chart U_1 for $v = 0$ we have the singular points $q_1 = (0,0)$ and $q_2 = \left(-\frac{e}{d}, 0\right)$ of system (5.17). The eigenvalues associated to q_1 and q_2 are $\lambda_{11} = e$, $\lambda_{12} = 0$ and $\lambda_{21} = -e$, $\lambda_{22} = 0$, respectively. By Theorem 1 of page 151 of [19], we have that q_1 is a topological saddle if $e > 0$, and stable node if $e < 0$. Analogously after the change of variables $(u, v, t) \mapsto \left(x - \frac{f}{d}y - \frac{e}{d}, y, -\frac{s}{e}\right)$ in system (5.17), we obtain that q_2 is a topological saddle if $e < 0$, and a stable node if $e > 0$.

For case $e = 0$ in the chart U_1 we have that $q_1 = q_2 = (0,0)$ is a linearly zero singular point. We do a blow-up in the direction u. More precisely, doing $u = \tilde{x}$

and $v = \tilde{x}\tilde{y}$ in system (5.17) and after a time rescaling, we obtain

$$\tilde{x}' = d\tilde{x} + f\tilde{x}\tilde{y} - \tilde{x}^2\tilde{y}^2, \qquad \tilde{y}' = -d\tilde{y} - f\tilde{y}^2. \tag{5.19}$$

When $f \neq 0$, system (5.19) has two singularities $\tilde{q}_1 = (0,0)$ and $\tilde{q}_2 = \left(0, -\dfrac{d}{f}\right)$ with respective eigenvalues $\tilde{\lambda}_1 = d$, $\tilde{\mu}_1 = -d$, $\tilde{\lambda}_2 = 0$ and $\tilde{\mu}_2 = d$. Note that $\tilde{q}_1$ is always a saddle. Now doing the change of variables $(\tilde{x}, \tilde{y}, t) \mapsto \left(\tilde{u}, \tilde{v} - \dfrac{d}{f}, \dfrac{s}{d}\right)$ to system (5.19), it becomes

$$\tilde{u}' = -\frac{d}{f^2}\tilde{u}^2 + \frac{f}{d}\tilde{u}\tilde{v} + \frac{2}{f}\tilde{u}^2\tilde{v} - \frac{1}{d}\tilde{u}^2\tilde{v}^2, \qquad \tilde{v}' = \tilde{v} - \frac{f}{d}\tilde{v}^2.$$

Hence by Theorem 1 of page 151 of [19], we have that $\tilde{q}_2$ is a saddle-node.

Now when $f = 0$, $\tilde{q}_1$ is the unique singularity of system (5.19), and as the previous case it is a saddle.

We do a blow-up in the direction v. More precisely, doing $u = \tilde{x}\tilde{y}$ and $v = \tilde{y}$ in system (5.17) and after a time rescaling, we obtain

$$\tilde{x}' = f\tilde{x} + d\tilde{x}^2, \qquad \tilde{y}' = -\tilde{y}^2. \tag{5.20}$$

We have to study only the singular point $\tilde{q}_3 = (0,0)$ of system (5.20). This singular point has eigenvalues $\tilde{\lambda}_3 = f$ and $\tilde{\mu}_3 = 0$, and so $\tilde{q}_3$ is a saddle-node, by Theorem 1 of page 151 of [19], if $f \neq 0$.

If $f = 0$ we do a new blow-up to system (5.20) in the direction $\tilde{x}$ ($\tilde{x} = \tilde{u}$ and $\tilde{y} = \tilde{u}\tilde{v}$) obtaining, after a time rescaling

$$\tilde{u}' = d\tilde{u}, \qquad \tilde{v}' = -d\tilde{v} - \tilde{v}^2. \tag{5.21}$$

System (5.21) has two singular points $(0,0)$ and $(0,-d)$ with respective eigenvalues $[d,-d]$ and $[d,d]$, so $(0,0)$ is a saddle, and $(0,-d)$ is a node (stable if $d < 0$ and unstable if $d > 0$).

Now doing a blow-up in direction $\tilde{y}$ ($\tilde{x} = \tilde{u}\tilde{v}$ and $\tilde{y} = \tilde{v}$), system (5.20) becomes after time rescaling

$$\tilde{u}' = \tilde{u} + d\tilde{u}^2, \qquad \tilde{v}' = -\tilde{v}. \tag{5.22}$$

We have that $(0,0)$ is a saddle of system (5.22).

Going back through the blow-ups we conclude that q_1 is either a saddle-node if $f \neq 0$, or a singular point with two hyperbolic sectors and two parabolic sectors if $f = 0$. □

5.5. **Case 5: $p(x)$ has two complex conjugated roots.** In this case we can write system (1.3) as

$$\dot{x} = x^2 - 2\alpha x + \alpha^2 + \beta^2, \qquad \dot{y} = dy^2 + (ex + f)y. \tag{5.23}$$

Note that $\alpha \pm i\beta$ are the roots of $x^2 - 2\alpha x + \alpha^2 + \beta^2 = 0$, and so system (5.23) does not have finite singular points. The next result determine the local phase portrait of the infinite singular points.

Proposition 5.10. *Let $p(X)$ be the Poincaré compactification of system* (5.23).

(a) *If $e \neq 1$, then $p(X)$ has six singularities $\pm q_1$, $\pm q_2$ and $\pm q_3$ in the equator $\mathbb{S}^1$. Moreover, q_1 (resp. q_2) is either a saddle (resp. stable node) if $e > 1$, or a stable node (resp. saddle) if $e < 1$, and q_3 is stable (resp. unstable) node when $d > 0$ (resp. $d < 0$).*

(b) *If $e = 1$, then $p(X)$ has four singularities $\pm q_1$ and $\pm q_3$ in the equator $\mathbb{S}^1$. Moreover q_1 is a saddle-node, and q_3 is either a stable node when $d > 0$, or an unstable node when $d < 0$.*

Proof. The system associated to $p(X)$ in the charts U_1 and U_2 are

$$\begin{array}{rcl} u' & = & (e-1)u + du^2 + (2\alpha + f)uv - (\alpha^2 + \beta^2)uv^2, \\ v' & = & -v + 2\alpha v^2 - (\alpha^2 + \beta^2)v^3, \end{array} \tag{5.24}$$

and

$$\begin{array}{rcl} u' & = & -du + (1-e)u^2 - (2\alpha + f)uv + (\alpha^2 + \beta^2)v^2, \\ v' & = & -dv - euv - fv^2, \end{array} \tag{5.25}$$

respectively.

In the chart U_2, $q_3 = (0,0)$ is a singular points of system (5.25), and its eigenvalues are $\lambda_{31} = \lambda_{32} = -d$. Therefore q_3 is stable (resp. unstable) node when $d > 0$ (resp. $d < 0$).

We suppose $e \neq 1$. In the chart U_1 for $v = 0$ we have the singular points $q_1 = (0,0)$ and $q_2 = \left(\dfrac{1-e}{d}, 0\right)$ of system (5.24). The eigenvalues associated to q_1 and q_2 are $\lambda_{11} = e - 1$, $\lambda_{12} = -1$ and $\lambda_{21} = 1 - e$, $\lambda_{22} = -1$, respectively. Therefore, the proof of statement (a) follows from the signs of the eigenvalues.

For case $e = 1$ in the chart U_1 we have that $q_1 = q_2 = (0,0)$, and the eigenvalues are $\lambda_{11} = 0$, $\lambda_{12} = -1$. By Theorem 1 of page 151 of [19], we have that q_1 is a saddle-node. Hence statement (b) follows. □

6. Main Results

In this section we classify all global phase portraits in the Poincaré disk of system (1.3). The first result is about the existence of limit cycles.

Proposition 6.1. *Systems* (1.3) *do not have limit cycle.*

Proof. Observe that the first equation of system (1.3) does not depend of the variable y. Hence, solving this differential equation, the solutions are not a periodic functions and so system (1.3) does not have periodic solutions. □

Theorem 6.2. *Consider system* (1.3). *If $p(x) = ax^2 + bx + c$ has two distinct reals roots, then the phase portrait is topological equivalent to one of the phase portraits of Figure 1.*

Proof. In this case system (1.3) can be written in the form (5.1). We have that $x = \alpha$, $x = \beta$ and $y = 0$ are invariant straight lines of system (5.1). These three straight lines intersect in the singular points $p_1 = (\alpha, 0)$ and $p_2 = (\beta, 0)$, and determine four infinite singular points $\pm q_1$ and $\pm q_3$ corresponding to the origin of the charts U_1, V_1, U_2 and V_2 in the Poincaré compactification, respectively. By Theorem 5.4, $\pm q_3$ are always a nodes. Moreover we can have additionally two infinite singular points $\pm q_2$ and either one, or two finite singular points p_3 and p_4.

First we suppose system (5.1) has four finite singular points. By Theorem 5.1, p_1 and p_2 are saddles or nodes.

When they are saddles, p_3 and p_4 are nodes and, by statements (3) and (6) of Theorem 5.1, these nodes live in opposite half-planes determined by the invariant straight line $y = 0$, and we obtain that $1 - e > 0$. In fact, consider the statements (3) of Theorem 5.1, we have that $e\alpha + f < 0$ and $-e\beta - f < 0$ and so $e(\alpha - \beta) < 0$.

Now, as $\alpha - \beta > 0$, it follows that $(\alpha - \beta) - e(\alpha - \beta) = (\alpha - \beta)(1 - e) > 0$, i.e. $1 - e > 0$. Since $1 - e > 0$, by Theorem 5.4, we always have six infinite singular points, $\pm q_1$ are nodes and $\pm q_2$ are saddles. Therefore in this case using Theorem 4.2 the phase portrait of system (5.1) is equivalent to Figure 1 (a).

If p_1 and p_2 are nodes, as in the previous case, p_3 and p_4 are saddles and live in opposites half-planes determined by the invariant straight line $y = 0$. However in this case we can have $1 - e \neq 0$ and $1 - e = 0$. Hence, by Theorem 5.4, there are either six infinite singular points (i.e., $\pm q_1$ and $\pm q_2$ are nodes or saddles), or four infinite singular points (i.e., $\pm q_1$ are saddle-nodes). Thus the phase portrait of system (5.1) is equivalent to one of Figure 1 (b)-(c).

If p_1 is saddle (resp. node) and p_2 is node (resp. saddle), then by statements (1), (4), (5) and (8) of Theorem 5.1, p_3 is a node (resp. a saddle) and p_4 is a saddle (resp. a node) and they live in the same half-plane determined by the invariant straight line $y = 0$. Moreover as in the previous case there are either six infinite singular points (i.e., $\pm q_1$ and $\pm q_2$ are nodes or saddles), or four infinite singular points (i.e., $\pm q_1$ are saddle-nodes). Note that when $\pm q_1$ is a saddle, we have a heteroclinic connection between a finite saddle and $\pm q_1$. Otherwise we do not have heteroclinic orbits. Thus in this case the phase portrait of system (5.1) is equivalent to one of Figure 1 (d)-(f).

Suppose system (5.1) has three finite singular points. By Theorem 5.2 these singular points are a saddle-node, a saddle and a node. Moreover the saddle-node is p_1 or p_2. If we have a saddle in the variant straight line $y = 0$, then by statements (1), (4), (6) and (7) of Theorem 5.2 and by Theorem 5.4, the infinite singular points $\pm q_1$ are nodes and $\pm q_2$ are saddles. Thus the phase portrait is equivalent to Figure 1 (i).

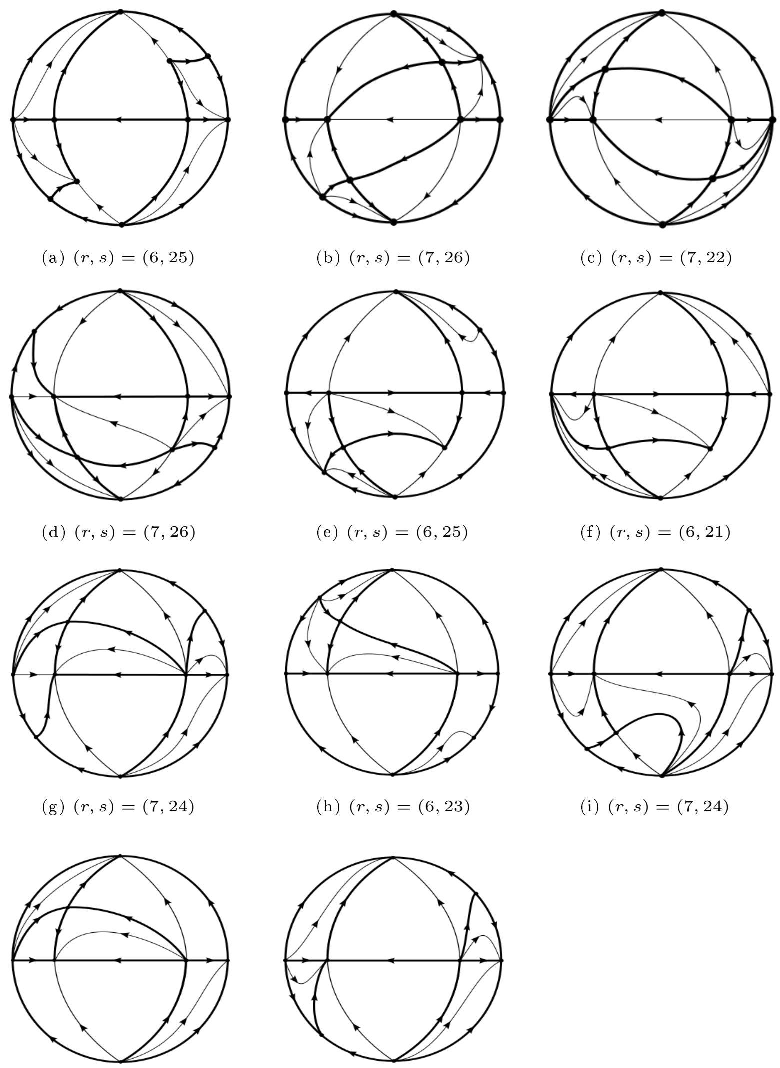
Now if we have a node in the invariant straight line $y = 0$, then by statements (2), (3), (5) and (8) of Theorem 5.2 and by Theorem 5.4, we can have either four or six infinite singular points. When there exist only four infinite singular points $\pm q_1$ are saddle-nodes and the phase portrait is equivalent to Figure 1 (j). When there exist six infinite singular points and $\pm q_1$ are nodes (resp. saddles), then $\pm q_2$ are saddles (resp. nodes) and phase portrait is equivalent to one of Figure 1 (g)-(h).

Finally we consider the case that system (5.1) has two finite singular points. By Theorem 5.3 these singular points are saddle-nodes. Now as $e\alpha + f = e\beta + f = 0$ and $\alpha \neq \beta$, we obtain $e = 0$. Hence by Theorem 5.4 system (5.1) has six infinite singular points, $\pm q_1$ and $\pm q_3$ are nodes and $\pm q_2$ are saddles, then the phase portrait is equivalent to Figure 1 (k). □

Theorem 6.3. *Consider system* (5.5). *If* $p(x)$ *have one real double root, then the phase portraits are topological equivalent to one of Figure 2.*

Proof. In this case system (1.3) can be written in the form (5.5). We have that $x = \alpha$ and $y = 0$ are invariant invariant straight lines of system (5.5). These straight lines intersect at the singular point $p_1 = (\alpha, 0)$ and determine four infinite singular points $\pm q_1$ and $\pm q_3$ corresponding to the origin of the charts U_1, V_1, U_2 and V_2 in the Poincaré compactification. By Theorem 5.4 $\pm q_3$ are always nodes. Moreover we can have additionally two infinite singular points $\pm q_2$, and one finite singular point p_3.

By Proposition 5.5 if $e\alpha + f \neq 0$, we have two finite singular points, both are saddle-nodes. If $1 - e \neq 0$, by Theorem 5.4, we have six infinite singular points.



(a) $(r, s) = (6, 25)$ (b) $(r, s) = (7, 26)$ (c) $(r, s) = (7, 22)$

(d) $(r, s) = (7, 26)$ (e) $(r, s) = (6, 25)$ (f) $(r, s) = (6, 21)$

(g) $(r, s) = (7, 24)$ (h) $(r, s) = (6, 23)$ (i) $(r, s) = (7, 24)$

(j) $(r, s) = (6, 19)$ (k) $(r, s) = (6, 21)$

FIGURE 1. Phase Portraits of Case 1. Here r denotes the number of canonical regions of the phase portrait and s its number of separatrices.

When $1 - e < 0$, $\pm q_1$ are saddles, $\pm q_2$ are nodes and we have a connection between the separatrices of a hyperbolic sector from p_1 with one of these infinite saddles and the phase portrait is topologically equivalent to Figure 2 (a). Now if $1 - e > 0$, $\pm q_1$ are nodes, $\pm q_2$ are saddles, and the phase portrait are topologically equivalent to Figure 2 (b). For $1 - e = 0$ by Theorem 5.4 we have four infinite singular points and $\pm q_1$ are saddle-nodes, so the phase portrait is topologically equivalent to Figure 2 (c).

In the case $e\alpha + f = 0$ by Theorem 5.5, p_1 is the only finite singular point and it is a singular point with two parabolic sectors and two hyperbolic sectors. Now by Theorem 5.4, we have six infinite singular points when $1 - e \neq 0$ and four infinite singular points otherwise. Hence the phase portrait is equivalent to one of Figure 2 (d)-(e). □

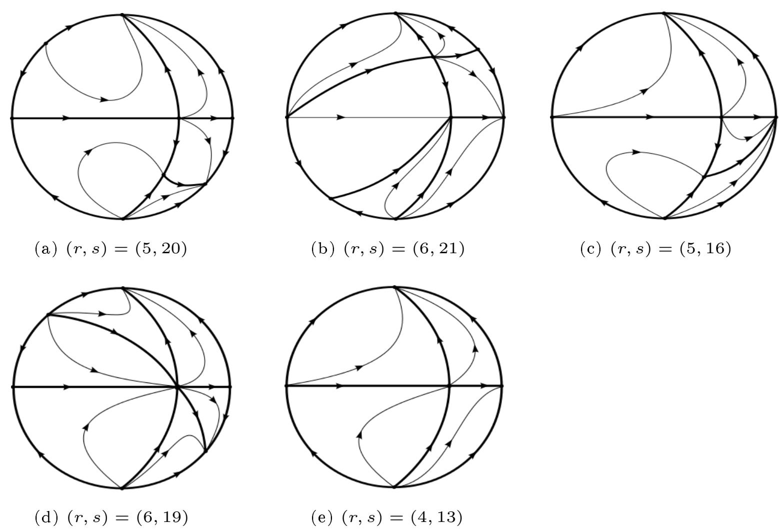


FIGURE 2. Phase Portraits of Case 2.

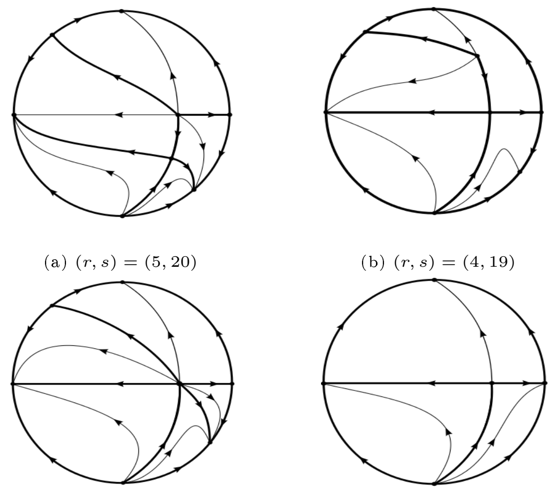
Theorem 6.4. *Consider system* (5.10)*. If* $p(x)$ *have a unique real root, then the phase portraits are topological equivalent to one of Figure 3.*

Proof. In this case system (1.3) can be written in to the form (5.10). We have that $x = \alpha$ and $y = 0$ are invariant straight lines of system (5.10). These straight lines intersect at the singular point $p_1 = (\alpha, 0)$ and determine four infinite singular points $\pm q_1$ and $\pm q_3$ corresponding to the origin in the charts U_1, V_1, U_2 and V_2 in the Poincaré compactification, respectively. By Theorem 5.8 q_3 is always a node. Moreover we can have additionally two infinite singular points $\pm q_2$ and one finite singular point p_2.

By Theorems 5.7 and 5.8 if $e\alpha + f > 0$ and $e \neq 0$, then p_1 is a node, p_2 is a saddle, $\pm q_1$ and $\pm q_2$ are saddle-nodes. Hence the phase portrait is topologically equivalent to Figure 3 (a). Analogously if $e\alpha + f < 0$, p_1 is a saddle, p_2 is a node and the phase portrait is topologically equivalent to Figure 3 (b).

If $e\alpha + f = 0$, then there exist a unique finite singular point p_1, and by Theorem 5.7 it is a saddle-node. When $e \neq 0$ by Theorem 5.8 we have six infinite singular points, the saddle-nodes $\pm q_1$ and $\pm q_2$ and the nodes $\pm q_3$. Then the phase portrait is topologically equivalent to Figure 3 (c). Now when $e = 0$, by Theorem 5.8, we have four infinite singular points, i.e., $\pm q_1$ are singular points with two hyperbolic sectors and two parabolic sectors and $\pm q_3$ are nodes. Hence the phase portrait is given by Figure 3 (d). □

Theorem 6.5. *Consider system* (5.16)*. Then the phase portraits are topological equivalent to one of Figure 4.*



(a) $(r,s) = (5,20)$ (b) $(r,s) = (4,19)$

(c) $(r,s) = (5,18)$ (d) $(r,s) = (3,12)$

FIGURE 3. Phase Portraits of Case 3.

Proof. In this case system (1.3) can be written in to the form (5.16). We have that $y = 0$ is an invariant straight line of system (5.16). This straight line determines two infinite singular points $\pm q_1$ corresponding to the origin of the charts U_1 and V_1 in the Poincaré compactification, respectively. In this case, we do not have finite singular points and by Theorem 5.9, the singular points $\pm q_3$, corresponding to the origin of the charts U_2 and V_2, always are nodes. Moreover, when $e \neq 0$ we have six infinite singular points, i.e., we have additionally two infinite singular points $\pm q_2$. If $e > 0$, then $\pm q_1$ are topological saddles and $\pm q_2$ are nodes. For $e < 0$, $\pm q_1$ are nodes and $\pm q_2$ are topological saddles. Hence the phase portrait is topologically equivalent to one of Figure 4 (a)-(b).

Now when $e = 0$ by Theorem 5.9, we have four infinite singular pints. Moreover, $\pm q_1$ are saddle-nodes if $f \neq 0$, or singular points with two hyperbolic sectors and two parabolic sectors if $f = 0$. Therefore the phase portrait s topologically equivalent to one of Figure 4 (c)-(d). □

Theorem 6.6. *Consider system* (5.23). *Then the phase portraits are topological equivalent to one of Figures 4 (a), (b) and (d).*

Proof. The proof of this theorem is analogous to Theorem 6.5. However in this case we do not have an infinite singular point with two parabolic and two hyperbolic sectors, and so we have only three phase portraits given in Figures 4 (a), (b) and (d). □

Proof of Theorem 1: The proof of Theorem 1 it follows from Theorems 6.3, 6.4, 6.5 and 6.6. By Theorem 4.2, phase portraits with distinct numbers (r, s) are not topologically equivalent. Note that (r, s) are distinct in all Figures 1–4, except in Figures 1 (a) and (e); Figures 1 (b) and (d); Figures 1 (f), (k) and Figure 2 (b); Figures 1 (g) and (i); Figure 1 (j) and Figure 2 (d); Figure 2 (a) and Figure 3 (a).

The phase portraits of Figures 1 (a) and (e) are topologically distinct, because in (a) we have a saddle connection between the finite saddles and in (e) do not. The

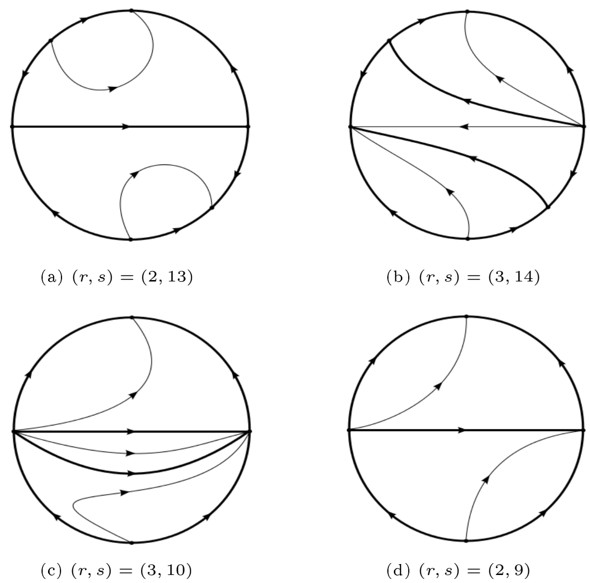


FIGURE 4. Phase Portraits of Case 4.

phase portraits in Figures 1 (b) and (d) are topologically equivalent by Theorem 4.2. The phase portrait of Figure 1 (f) is topologically distinct of Figures 1 (k) and Figure 2 (b), because Figure 1 (f) we have only four infinite singular points. Now, doing a rotation by a angle of $\pi/2$ radians, after a reflection through the y-axis and reversing the orientation of the orbits, is easy to see that the phase portraits of Figures 1 (k) and Figure 2 (b) are topologically equivalent. The phase portraits of Figure 1 (g) and (i) are topologically distinct, because in Figure 1 (i) we have a connection between a finite and infinite saddle, and in Figure 1 (g) do not. The phase portraits of Figure 1 (j) and Figure 2 (d) are topologically distinct, because in Figure 1 (j) we have three finite singular points and in Figure 2 (d) we have one finite singular point. The phase portraits of Figure 2 (a) and Figure 3 (a) are topologically distinct, because in Figure 2 (a) the finite singular points are two saddle-nodes and in Figure 3 (a) the finite singular points are a node and a saddle. □

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