GLOBAL PHASE PORTRAITS OF THE QUADRATIC SYSTEMS HAVING A SINGULAR AND IRREDUCIBLE INVARIANT CURVE OF DEGREE 3

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ABSTRACT. Any singular irreducible cubic after an affine transformation can be written as either $y^2 = x^3$, or $y^2 = x^2(x+1)$, or $y^2 = x^2(x-1)$. We classify the phase portraits of all quadratic polynomial differential systems having the invariant cubic $y^2 = x^2(x+1)$. We prove that there are 65 different topological phase portraits for such quadratic polynomial differential systems.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Quadratic polynomial differential systems (or simply *quadratic systems*) are systems that can be written into the form

$$\dot{x} = P(x,y) = P_0 + P_1 + P_2, \qquad \dot{y} = Q(x,y) = Q_0 + Q_1 + Q_2, \tag{1}$$

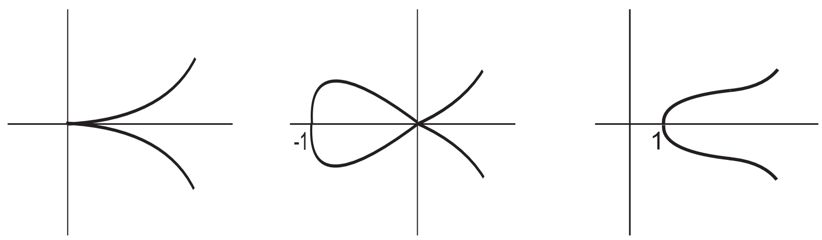
where P_i and Q_i are real polynomials of degree i in the variables (x,y) and $P_2^2 + Q_2^2 \neq 0$.

An extensive literature is dedicated to the study of the quadratic systems these last years. For a good survey see the book of Reyn [37] or the book of Artes et.al [4], and references therein. For example, the following families of quadratic systems have been studied: homogeneous [16], semi-homogeneous [12], bounded [18], reversible [24, 17], Hamiltonian [2, 15], Lienard [19], integrable using Carleman and Painlevé tools [25], rational integrable [5, 6, 7], the ones having a star nodal point [10], a center [42, 31, 17, 42], one focus and one antisaddle [3], with a semi–elementary triple node [8], chordal [21, 22], with four infinite singular points and one invariant straight line [38], with invariant lines [41], and so on. There is also an extensively literature about Hilbert's sixteen problem and quadratic systems, see for example [13, 23, 29, 30, 31, 44], and the notion of ciclicity [45, 26, 15], and so on. For the study of some geometric properties of quadratic systems see [39, 40], and others. In particular we pay attention on reference [28] where the authors present a classification of all quadratic systems having one real reducible invariant algebraic curve of degree 3.

In [11] it is proved that a cubic algebraic curve (or simply a cubic) is singular and irreducible if and only if it can be written after affine transformations into one of the forms

$$y^2 = x^3, \qquad y^2 = x^2(x+1), \qquad y^2 = x^2(x-1).$$

See Figure 1.



$f_1 = y^2 - x^3$ $\qquad$ $f_2 = y^2 - x^2(x+1)$ $\qquad$ $f_3 = y^2 - x^2(x-1)$

FIGURE 1. Singular and irreducible algebraic curves of degree 3.