ON THE NUMBER OF LIMIT CYCLES IN GENERALIZED ABEL EQUATIONS

JIANFENG HUANG, JOAN TORREGROSA, AND JORDI VILLADELPRAT

Abstract. Given $p, q \in \mathbb{Z}_{\geq 2}$ with $p \neq q$, we study generalized Abel differential equations

$$\frac{dx}{d\theta} = A(\theta)x^p + B(\theta)x^q,$$

where A and B are trigonometric polynomials of degrees $n, m \geq 1$, respectively, and we are interested in the number of limit cycles (i.e., isolated periodic orbits) that can have. More concretely, in this context an open problem is to prove the existence of an integer, depending only on p, q, m, and n and that we denote by $\mathcal{H}_{p,q}(n, m)$, such that the above differential equation has at most $\mathcal{H}_{p,q}(n, m)$ limit cycles. In the present paper, by means of a second order analysis using Melnikov functions, we provide lower bounds of $\mathcal{H}_{p,q}(n, m)$ that, to the best of our knowledge, are larger than the previous ones appearing in the literature. In particular, for classical Abel differential equations (i.e., $p = 3$ and $q = 2$), we prove that $\mathcal{H}_{3,2}(n, m) \geq 2(n + m) - 1$.

1. Introduction and statements of main results

The study of the existence of periodic orbits in ordinary differential equations has been an interesting problem for years in many areas of mathematics, particularly in qualitative theory of differential equations. In this area of interest, when we focus on planar polynomial vector fields, one of the most renowned classical problems arises: to know the number and location of isolated periodic orbits, the so-called *limit cycles*, in terms of its degree n. The study of this problem began at the end of the 19th century with the seminal works by Poincaré, but takes its name after Hilbert because of his famous list of unsolved problems published in 1900. From the original list of 23 problems, the 16th is still open, in particular its second part. More precisely (see [26] or [36] for details), the 'existential' Hilbert's 16th problem is to prove that for any $n \geq 2$ there exists a finite number $\mathcal{H}(n)$ such that any polynomial vector field of degree $\leq n$ has less than $\mathcal{H}(n)$ limit cycles.

Motivated by 16th Hilbert's problem, a very related line of research is to investigate the periodic solutions of scalar differential equations

$$\dot{x} := \frac{dx}{d\theta} = \sum_{i=0}^{k} a_i(\theta)x^i,$$

where a_i are periodic analytic functions. In this context an isolated periodic solution is called limit cycle and it occurs that its number increases with k. (The reader is referred to [15] for an enlightening explanation of this fact.) Linear differential equations have at most 1 limit cycle, whereas the quadratic ones have at most 2. The latter are known as Riccati equations and the upper bound follows from the fact that the return map is a Möbius function. Nevertheless the situation is more intricate for degree three. The well-known trigonometric Abel differential equation writes as

$$\dot{x} = A(\theta)x^3 + B(\theta)x^2 + C(\theta)x, \tag{1}$$

being A, B, and C trigonometric polynomials. Pliss [33] proves using the Schwarzian derivative that if A does not change sign then the maximum number of limit cycles is 3. However it was Lins-Neto [31] the first to show that, in general, there is no upper bound for the number of limit cycles. Indeed, he proves that for every positive integer ℓ there exists an Abel differential

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