Canards existence in the Hindmarsh–Rose model

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1 Introduction

The concept of "canard solutions" for three-dimensional singularly perturbed systems with two *slow* variables and one *fast* has been introduced in the beginning of the eighties by Benoît and Lobry [**2**], Benoît [**3**]. Their existence has been proved by Benoît [**3**, p. 170] in the framework of "Non-Standard Analysis" according to a theorem which states that canard solutions exist in such systems provided that the *pseudo singular point* of the *slow dynamics*, *i.e.*, of the *reduced vector field* is of *saddle* type. Nearly twenty years later, Szmolyan and Wechselberger [**12**] provided a "standard version" of Benoît's theorem [**3**]. Recently, Wechselberger [**15**] generalized this theorem for n-dimensional singularly perturbed systems with k *slow* variables and m *fast* (where $n = k + m$). The method they used require to implement a "desingularization procedure" which can be summarized as follows: first, they compute the *normal form* of such singularly perturbed systems which is expressed according to some coefficients (a and b for dimension three and $\tilde{a}$, $\tilde{b}$ and $\tilde{c}_1$ for dimension four) depending on the functions defining the original vector field and their partial derivatives with respect to the variables. Secondly, they project the "desingularized vector field" (originally called "normalized slow dynamics" by Eric Benoît [**3**, p. 166]) of such a *normal form* on the tangent bundle of the critical manifold. Finally, they evaluate the Jacobian of the projection of this "desingularized vector field" at the *folded singularity* (originally called *pseudo singular points* by José Argémi [**1**, p. 336]). This lead Szmolyan and Wechselberger [**12**, p. 427] and Wechselberger [**15**, p. 3298] to a "classification of *folded singularities* (*pseudo singular points*)". Thus, they showed that for three-dimensional (resp. four-dimensional) singularly perturbed systems such *folded singularity* is of *saddle type* if the following condition is satisfied: $a < 0$ (resp. $\tilde{a} < 0$).

In a first paper entitled: "Canards Existence in Memristor's Circuits" (see Ginoux & Llibre [**4**]) we presented a method enabling to state a unique "generic" condition for the existence of "canard solutions" for three and four-dimensional singularly perturbed systems with only one fast variable which is based on the stability of *folded singularities* of the *normalized slow dynamics* deduced from a well-known property of linear algebra. We proved that this unique condition is completely identical to that provided by Benoît [**3**], Szmolyan and Wechselberger [**12**] and Wechselberger [**15**].

In a second paper entitled: "Canards Existence in FitzHugh-Nagumo and Hodgkin-Huxley Neuronal Models" (see Ginoux & Llibre [**5**]) we extended this method to the case of four-dimensional singularly perturbed systems with $k = 2$ *slow* and $m = 2$ *fast* variables. Then, we stated that the provided condition for the existence of canards is "generic" since it is exactly the same for singularly perturbed systems of dimension three and four with one or two *fast* variables. The method we used led us to the following proposition: *If the normalized slow dynamics has a pseudo singular point of saddle type, i.e. if the sum σ_2 of all second-order diagonal minors of the Jacobian matrix of the normalized slow dynamics evaluated at the pseudo singular point is negative, i.e. if $\sigma_2 < 0$ then, the three-dimensional (resp. four-dimensional) singularly perturbed system exhibits a canard solution which evolves from the attractive part of the slow manifold towards its repelling part.* Then, we proved on one hand for three-dimensional singularly perturbed systems with only one fast variable that the condition for which the *pseudo singular point* is of saddle type, *i.e.* $\sigma_2 < 0$ is identical to that proposed by Benoît [**3**, p. 171] in his theorem, *i.e.* $D < 0$ and also to that provided by Szmolyan and Wechselberger [**12**], *i.e.* $a < 0$. On the other hand, we proved for four-dimensional singularly perturbed systems with one or two fast variables that the condition for which the *folded singularity* (resp. the *pseudo singular point*) is of saddle type, *i.e.* $\sigma_2 < 0$ is identical to that proposed by Wechselberger [**15**, p. 3298] in his theorem, *i.e.* $\tilde{a} < 0$.

Notice that there is no proof of the approximation. It is not established that the time–scaled reduced system holds on the approximation for the original system in the case of k slow variables ($k \geq 3$), m fast variables ($m \geq 2$). It was proved in the case $k = 2$ and $m = 1$ by Benoît; constructing a local model and obtaining its solution, and in the case $k = 2$ and $m = 2$ was also proved extensively by Tchizawa [**13, 14**]). For the case $k = 1$ and $m = 2$ (Hindmarsch–Rose model), we shall construct a local model again and we shall obtain their solutions, providing a constructive proof for the approximation. Being the pseudo–singular point a saddle, or a node it does not ensure the existence of canards, because it may not satisfy the approximation.

The aim of this work is to extend this method to the case of three-dimensional singularly perturbed systems with one *slow* and two *fast* variables and to show that the provided condition for the existence of canards, *i.e.* $\sigma_2 < 0$ still holds and is consequently "generic".

The Hindmarsh-Rose model [**8**] describes the basic properties of individual neurons and appears as a reduction of the conductance based in the Hodgkin-Huxley model for neural spiking, see for more details [**9**]. Thus, the three-dimensional Hindmarsh-Rose polynomial ordinary differential system was originally written as:

$$
\begin{aligned}
\frac{dx}{dt} &= y - ax^3 + bx^2 - z + I, \\
\frac{dy}{dt} &= c - dx^2 - y, \\
\frac{dz}{dt} &= r\left[s\left(x - \alpha\right) - z\right],
\end{aligned}
\tag{1}
$$

where x is a transmembrane neuron potential, y and z are the characteristics of ionic currents dynamic, I is ambient current. The other parameters (a, b, c, d, I, s, α and r) reflect the physical features of the neurons and the dot indicates derivative with respect to the time t. We notice that the parameter $r << 1$. Existence of canard solutions in such system (1) has been originally suspected by Shilnikov *et al.* [**10**, p. 2149] and highlighted by Shchepakina [**11**]. Thus, according to the previous definitions, the Hindmarsh-Rose model may be written as a three-dimensional singularly perturbed system with $k = 1$

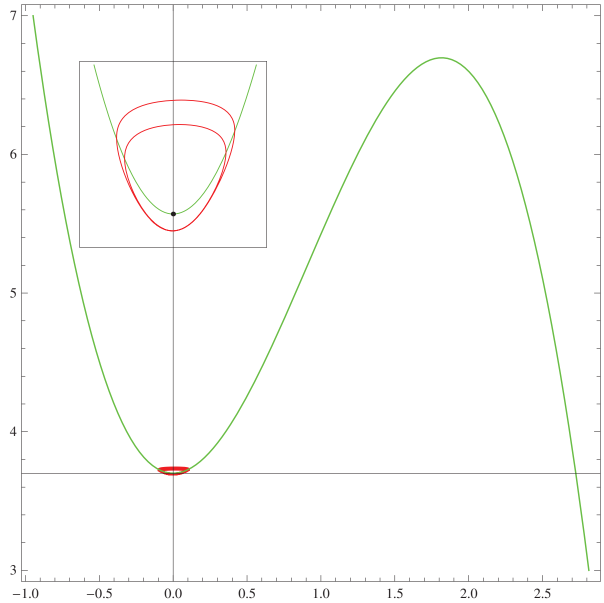

FIGURE 1. Canard solution of the Hindmarsh-Rose (1) model in the (x, z) plane phase with the following parameter set: $a = 1$, $b = 3$, $c = 1$, $d = 0.275255$, $I = 2.7$, $\alpha = -1.2$ and for the "duck parameter" value $s = 3.08104454785558141214$.

slow variable and $m = 2$ *fast* variables. By posing $x \to y_2$, $y \to y_1$, $z \to x_1$ and $t' \to \varepsilon t$ with $\varepsilon = r$, we obtain:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, y_1, y_2) = s(y_2 - \alpha) - x_1, \\ \varepsilon \dot{y}_1 &= g_1(x_1, y_1, y_2) = c - dy_2^2 - y_1, \\ \varepsilon \dot{y}_2 &= g_2(x_1, y_1, y_2) = y_1 - ay_2^3 + by_2^2 - x_1 + I, \end{aligned} \tag{2}$$

where $x_1 \in \mathbb{R}$, $\vec{y} = (y_1, y_2)^t \in \mathbb{R}^2$, $0 < \varepsilon \ll 1$ and the functions f_i and g_i are assumed to be C^2 functions of (x_1, y_1, y_2) and the dot now indicates derivative with respect to the time t'.

We have proved the existence of different kind of canard solutions for system (2) see Figures 1, 2 and 3.

In fact in the work Shchepakina [**11**] already was found the canard of Figure 1. We proved the existence of this canard showing the existence of a *pseudo singular point* of saddle-type when the parameters satisfy $s < (c + I)/\alpha$. With $c = 1$, $I = 2.7$ and $\alpha = -1.2$, we find that: $s < 3.0833$. Thus, Shchepakina highlighted a canard without head in the Hindmarsh-Rose model (see Fig. 1) for the "duck parameter" value $s = 3.08104454785558141214 < 3.0833$.

In the inset of Fig. 1, the zoom in highlights a large distance between the canard solution and that of the *critical manifold*. This is due to the fact that this latter corresponds to zero-order approximation in ε of the *slow invariant manifold*. Nevertheless, while using the so-called *Flow Curvature Method* Ginoux and Rossetto [**7**] have already provided a second-order approximation in ε of the *slow invariant manifold* of the Hindmarsh-Rose model (1). The result is presented in Fig. 2.

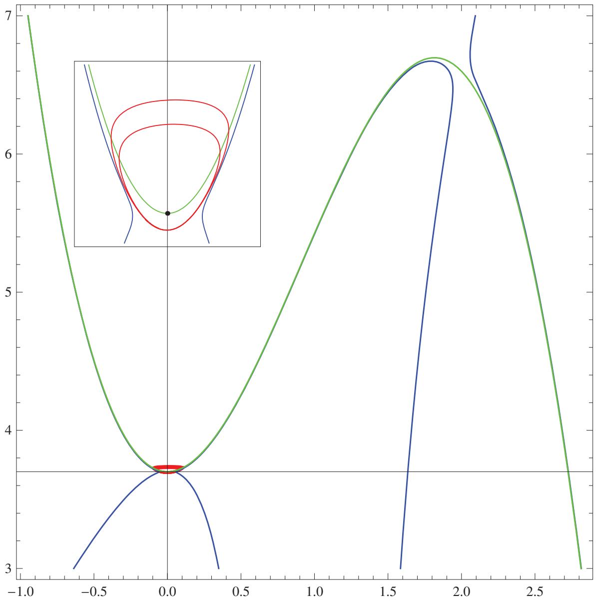

FIGURE 2. Canard solution of the Hindmarsh-Rose (1) model in the (x, z) plane phase, its *critical manifold* (in green) and the second-order approximation in ε of the *slow invariant manifold* (in blue) with the following parameter set: $a = 1$, $b = 3$, $c = 1$, $d = 0.275255$, $I = 2.7$, $\alpha = -1.2$ and for the "duck parameter" value $s = 3.08104454785581412 14$.

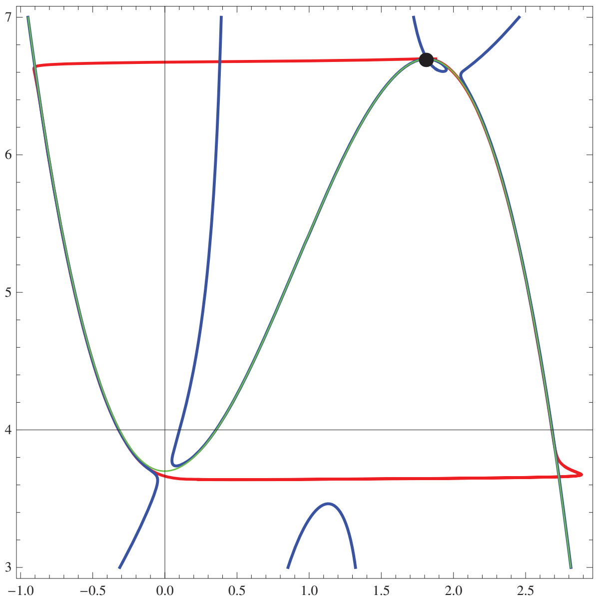

FIGURE 3. Canard solution of the Hindmarsh-Rose (1) model in the (x, z) plane phase, its *critical manifold* (in green) and the second-order approximation in ε of the *slow invariant manifold* (in blue) with the following parameter set: $a = 1$, $b = 3$, $c = 1$, $d = 0.275255$, $I = 2.7$, $\alpha = -1.2$ and for the "duck parameter" value $s = 2.220095$.

With $c = 1$, $I = 2.7$ and $\alpha = -1.2$, we find that: $s < 2.2200954$. Thus, we have highlighted a canard with head in the Hindmarsh-Rose model (see Fig. 3) for the "duck parameter" value $s = 2.220095 < 2.2200954$. For this parameters set the second-order approximation in ε of the *slow invariant manifold* of the Hindmarsh-Rose model (1) can be provided while using the *Flow Curvature Method* introduced by Ginoux and Rossetto [**7**]. The result is presented in Fig. 3.

All the details of the existence of these three different canards in the Hindmarsh-Rose model [**8**] can be found in [**6**].

Acknowledgments

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