HIGHEST WEAK FOCUS ORDER FOR TRIGONOMETRIC LIÉNARD EQUATIONS

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Abstract. Given a planar analytic differential equation with a critical point which is a weak focus of order k, it is well-known that at most k limit cycles can bifurcate from it. Moreover, in case of analytic Liénard differential equations this order can be computed as one half of the multiplicity of an associated planar analytic map. By using this approach, we can give an upper bound of the maximum order of the weak focus of pure trigonometric Liénard equations only in terms of the degrees of the involved trigonometric polynomials. Our result extends to this trigonometric Liénard case a similar result known for polynomial Liénard equations.

1. Introduction and main results

Recall that a critical point of a planar analytic vector field is called a *focus* if the eigenvalues of its linear approximation at the point are not real, i.e. $\alpha \pm \mathrm{i}\beta$, $\beta \neq 0$. Moreover, when $\alpha \neq 0$ the point is called a *strong focus* and, otherwise, it is called a *weak focus*. The complex Poincaré's normal form of its associated differential equation at this weak focus point is

$$\dot{z} = iz\Big(\beta + \sum_{j=1}^{\infty} c_j (z\bar{z})^{2j}\Big), \quad c_j = \alpha_j + \mathrm{i}\beta_j \in \mathbb{C}.$$

The values α_j give the so called *Lyapunov quantities* and can be computed in many other ways, see for instance [1, 4, 12, 16, 17, 18, 19, 23]. When all $\alpha_j = 0$ the weak focus is a *center*, otherwise, if $\alpha_k \neq 0$, is the first non-zero α_j then it is said that the origin is a *weak focus of order k*. It is well known that k is the maximum number of limit cycles (isolated periodic orbits) that bifurcate from this type of points, and that this amount of limit cycles is attained for some analytic perturbations. Therefore, given an analytic family, $\mathcal{F}$, of planar analytic differential equations depending on finitely many parameters, it is very interesting to know which is the maximum order of the weak focus inside this family, $\sigma(\mathcal{F})$, see [24]. This number is known to exist when the Lyapunov quantities are polynomials on the parameters of the system, because of the Hilbert's basis Theorem.

Unfortunately Hilbert's result is not constructive and, in general, an explicit bound of the number of needed Lyapunov quantities is not known. In fact, even for cubic vector fields this number is nowadays unknown.

The most important family $\mathcal{F}$ of systems of arbitrary degree for which an explicit upper bound of $\sigma(\mathcal{F})$ is known is the family of polynomial Liénard equations. This

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