TOPOLOGICAL PROPERTIES OF THE IMMEDIATE BASINS OF ATTRACTION FOR THE SECANT METHOD

LAURA GARDINI, ANTONIO GARIJO, AND XAVIER JARQUE

Abstract. We study the discrete dynamical system defined on a subset of R^2 given by the iterates of the secant method applied to a real polynomial p. Each simple real root α of p has associated its basin of attraction $\mathcal{A}(\alpha)$ formed by the set of points converging towards the fixed point (α, α) of S. We denote by $\mathcal{A}^*(\alpha)$ its immediate basin of attraction, that is, the connected component of $\mathcal{A}(\alpha)$ which contains (α, α). We focus on some topological properties of $\mathcal{A}^*(\alpha)$, when α is an internal real root of p. More precisely, we show the existence of a 4-cycle in $\partial\mathcal{A}^*(\alpha)$ and we give conditions on p to guarantee the simple connectivity of $\mathcal{A}^*(\alpha)$.

Keywords: Root finding algorithms, rational iteration, secant method, periodic orbits

MSC2010: 37G35, 37N30, 37C70

1. Introduction and statement of the results

Dynamical systems is a powerful tool in order to have a deep understanding on the global behavior of the so called *root-finding* algorithms, that is, iterative methods capable to numerically determine the solutions of the equation $f(x) = 0$. In most cases, it is well known the order of convergence of those methods near the zeros of f, but it is in general unclear the behavior and effectiveness when initial conditions are chosen on the whole space; a natural question when we do not know a priori where the roots are or if there are many of them.

The numerical exploration of the solutions of the equation $f(x) = 0$ has been always central problem in many areas of applied mathematics; from biology to engineering, since most mathematical models requires to have a thorough knowledge of the solutions of certain equations. Once we are certain that no algebraic manipulation of the equation will allow to explicitly find out the solutions, one can try to built numerical methods which will approximate the solutions with arbitrary precision. Perhaps the most well known and universal method is the *Newton method* inspired on the linearization of the equation $f(x) = 0$. But also other methods has shown to be certainly efficient like the *secant method*, the main object of the paper.

Roughly speaking, all these iterative methods give efficient ways to find the solutions of $f(x) = 0$, at least once you have a good approximation of them. However, there is a significant amount of uncertainty when the initial conditions are freely chosen, i.e. when there is not a *natural* candidate for the solution or the number of solutions is high. It is in this context where dynamical systems might play a central role. As an example we can refer to [HSS01] where the authors first prove theoretical results on the global dynamics of the Newton method and then apply them to create efficient algorithms to find out *all* solutions, even in the case that the degree of p is huge.

This work has been partially supported by MINECO-AEI grants MTM-2017-86795-C3-2-P and MTM-2017-86795-C3-3-P and the AGAUR grant 2017 SGR 1374.