The zero-Hopf bifurcations in the Kolmogorov systems of degree 3 in $\mathbb{R}^3$

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Abstract

In this work we study the periodic orbits which bifurcate from all zero-Hopf bifurcations that an arbitrary Kolmogorov system of degree 3 in $\mathbb{R}^3$ can exhibit. The main tool used is the averaging theory.

Keywords: Lotka–Volterra system, Kolmogorov systems, phase portraits, Hopf bifurcation, zero-Hopf bifurcation, limit cycle

1. Introduction and statement of the main results

Lotka-Volterra systems were initially proposed, independently, by Alfred J. Lotka in 1925 [1] and Vito Volterra in 1926 [2], both in the context of competing species. These Lotka-Volterra systems are polynomial differential systems of the form

$$\dot{x} = xP(x, y), \qquad \dot{y} = yQ(x, y),$$

where P and Q are polynomials of degree 1. Later on the *Lotka-Volterra systems* were generalized and considered on arbitrary dimension $n \geq 2$, i.e.

$$\dot{x}_i = x_i P_i(x_1, \ldots, x_n),$$

where P_i are polynomials of degree 1. Finally in 1936 Andrei Kolmogorov [3] extended those systems to arbitrary degree, i.e. the polynomials P_i can have any degree. These last systems are now called *Kolmogorov systems*.

The Lotka-Volterra and Kolmogorov systems have been used for modelling many natural phenomena, such as the time evolution of conflicting species in biology [4], chemical reactions [5], plasma physics [6], hydrodynamics [7], and many other phenomena as social science and economics [8]. Recently limit cycles for differential systems in $\mathbb{R}^3$ also are studied for discontinuous differential systems see for instance [9] and the references quoted therein.

We want to study the limit cycles of the Kolmogorov systems of degree 3 in $\mathbb{R}^3$ which bifurcate in the zero-Hopf bifurcations of the singular points (a, b, c) which are not on the invariant planes $x = 0$, $y = 0$ and $z = 0$ of the Kolmogorov system

$$\dot{x} = xP(x, y, z), \quad \dot{y} = yQ(x, y, z), \quad \dot{z} = zR(x, y, z),$$

with P, Q and R polynomials of degree 2. Doing the scaling $(x, y, z) \to (x/a, y/b, z/c)$ we can assume without loss of generality that $(a, b, c) = (1, 1, 1)$. Therefore it is sufficient to study the limit cycles which can bifurcate

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