TOPOLOGICALLY ANOSOV PLANE HOMEOMORPHISMS.

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ABSTRACT. This paper deals with classifying the dynamics of *Topologically Anosov* plane homeomorphisms. We prove that a Topologically Anosov homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ is conjugate to a homothety if it is the time one map of a flow. We also obtain results for the cases when the nonwandering set of f reduces to a fixed point, or if there exists an open, connected, simply connected proper subset U such that $U \subset Int(\overline{f(U)})$, and such that $\cup_{n\geq 0} f^n(U) = \mathbb{R}^2$. In the general case, we prove a structure theorem for the α-limits of orbits with empty ω-limit (or the ω-limits of orbits with empty α-limit), and we show that any basin of attraction (or repulsion) must be unbounded.

1. INTRODUCTION

A homeomorphism $f : M \to M$ of the metric space to itself is called *expansive* if there exists $\alpha > 0$ such that given $x, y \in M, x \neq y$, then $d(f^n(x), f^n(y)) > \alpha$ for some $n \in \mathbb{Z}$. The number α is called the *expansivity constant* of f.

The study of expansive systems is both classic and fascinating. In Lewowicz's words [8], the fact that every point has a distinctive dynamical meaning implies that a rich interaction between dynamics and topology is to be expected.

If $\delta > 0$, a δ-pseudo-orbit for f is a sequence $(x_n)_{n\in\mathbb{Z}}$ such that $d(f(x_n), x_{n+1}) < \delta$ for all $n \in \mathbb{Z}$. If $\epsilon > 0$, we say that the orbit of x ϵ-shadows a given pseudo-orbit if $d(x_n, f^n(x)) < \epsilon$ for all $n \in \mathbb{Z}$. Finally, we say that f has the shadowing property if for each $\epsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit is ϵ-shadowed by an orbit of f. In other words, systems with the shadowing property are precisely the ones in which "observational errors" do not introduce unexpected behavior, in the sense that simulated orbits actually "follow" real orbits.

Anosov diffeomorphisms, the best known chaotic dynamical systems, are expansive and have the shadowing property. Moreover, expansive homeomorphisms with the shadowing property on compact metric spaces are known to have spectral decomposition in Smale's sense ([1]).

On non-compact spaces however, it is well known that a dynamical system may be expansive or have the shadowing property with respect to one metric, but not with respect to another metric that induces the same topology. In [5] topological definitions of expansiveness and shadowing are given that are equivalent to the usual metric definitions for homeomorphisms on compact metric spaces, but are independent of any change of compatible metric. In [4], the author applies these definitions with the plane $\mathbb{R}^2$ as the phase space and proves a fixed point theorem. Following his spirit, we take these definitions and try to classify the dynamics with the plane $\mathbb{R}^2$ as the phase space.

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