A note on the Lyapunov and Period constants

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Abstract. It is well known that the number of small amplitude limit cycles that can bifurcate from the origin of a weak focus or a non degenerated center for a family of planar polynomial vector fields is governed by the structure of the so called Lyapunov constants, that are polynomials in the parameters of the system. These constants are essentially the coefficients of the odd terms of the Taylor development at zero of the displacement map. Although many authors use that the coefficients of the even terms of this map belong to the ideal generated by the previous odd terms, we have not found a proof in the literature. In this paper we present a simple proof of this fact based on a general property of the composition of one-dimensional analytic reversing orientation diffeomorphisms with themselves. We also prove similar results for the period constants. These facts, together with some classical tools like the Weirstrass preparation Theorem, or the theory of extended Chebyshev systems, are used to revisit some classical results on cyclicity and criticality for polynomial families of planar differential equations.

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1 Introduction and main results

Consider planar analytic vector fields $(x, y) \to F(x, y, \lambda) \in \mathbb{R}^2$ with $\lambda \in \mathbb{R}^m$ that have $(x, y) \to (-y, x)$ as their linearization at the origin. It is well known that for this type of vector fields the maximum number of small amplitude limit cycles that can bifurcate from the origin varying λ is governed by the the structure of the so called *Lyapunov constants*, that are polynomials in λ if the dependence on λ of F is as well polynomial. This number is called the *cyclicity* of the family ([1, 2, 9, 10]), see Section 2 for more details. In fact, the problem of determine the cyclicity, that can be seen as a multiple Hopf bifurcation, goes back to Bautin who in 1954 considered it for planar quadratic vector fields.

To be more precise, for any $n \geq 2$ and $i, j \geq 0$ such that $1 < i + j \leq n$, fix $u_{i,j}, v_{i,j} \in \mathbb{R}[\lambda_1, \ldots, \lambda_m]$ and let $\mathcal{F}$ be the family of polynomial vector fields given by

$$\mathcal{F} = \Big\{ F : F(x, y, \lambda) = \Big(-y + \sum_{i+j=2}^{n} u_{i,j}(\lambda) x^i y^j, x + \sum_{i+j=2}^{n} v_{i,j}(\lambda) x^i y^j \Big) \Big\}.$$

Clearly $\mathcal{F}$ is an m-parametric family of polynomial vector fields having a singularity of center or focus type at the origin. Note also that the dependence on the parameters is polynomial. For any $\lambda_0 \in \mathbb{R}^m$ we denote by F_{λ_0} the polynomial vector field obtained evaluating the polynomials $u_{i,j}, v_{i,j}$ at λ_0.

For $\lambda \in \mathbb{R}^m$ and $x > 0$ small enough let $\pi(\lambda, x)$ be the first intersection with the positive X-axis of the solution of the Cauchy's problem

$$(1) \qquad \begin{cases} (\dot{x}, \dot{y}) = F_\lambda(x, y), \\ x(0) = x,\ y(0) = 0. \end{cases}$$