DYNAMICS OF A FAMILY OF RATIONAL OPERATORS OF ARBITRARY DEGREE

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Abstract. In this paper we analyse the dynamics of a family of rational operators coming from a fourth-order family of root-finding algorithms. We first show that it may be convenient to redefine the parameters to prevent redundancies and unboundedness of problematic parameters. After reparametrization, we observe that these rational maps belong to a more general family $O_{a,n,k}$ of degree $n+k$ operators, which includes several other families of maps obtained from other numerical methods. We study the dynamics of $O_{a,n,k}$ and discuss for which parameters n and k these operators would be suitable from the numerical point of view.

1. Introduction

Iterative methods are the most usual tool to approximate solutions of non linear equations. These methods require at least one initial estimate close enough of the solution sought. It is known that the methods converge if the initial estimation is chosen suitably. Hence, the search of such initial conditions has became an important part in the study of iterative methods. To achieve this goal we analyse these methods as discrete dynamical systems.

The application of iterative methods to find solutions of equations of the form $f(z) = 0$, where $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, gives rise to discrete dynamical systems given by the iteration of rational functions. The best known numerical algorithm is Newton's method, whose dynamics has been widely studied (see for instance [6, 17]). Indeed, there are several results about the dynamical plane as well as the parameter plane of Newton's method applied to some concrete families of polynomials. The most studied case is Newton's method of cubic polynomials $q(z) = z(z-1)(z-\alpha)$, $\alpha \in \hat{\mathbb{C}}$ (see for instance [16, 19] and references therein).

Recently, this dynamical study has been enlarged to other numerical methods (see, for example, [9], [10], [11], [12], [13] and references therein). The dynamical properties related to an iterative method give important information about its stability. In recent studies, many authors (see [1], [7], [8], [13], [14], for example) have found interesting results from a dynamical point of view. One of the main interests in these papers has been the study of the parameter spaces associated to the families of iterative methods applied on low degree polynomials, which allows to distinguish the different dynamical behaviour.

In this paper, we consider an optimal fourth-order family of methods presented by R. Behl in [4], whose dynamics is partially studied by K. Argyros and A. Magreñán in [1]. The family of methods is given by

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= x_n - \frac{((b^2-22b-27)f'(x_n) + 3(b^2+10b+5)f'(y_n))f(x_n)}{2(bf'(x_n)+3f'(y_n))(3(b+1)f'(y_n)-(b+5)f'(x_n))}, \end{aligned}$$

where b is a complex parameter. When applying these methods on quadratic polynomials of the form $z^2 + c$ (compare with Section 2) we obtain an operator which is conjugate to

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