FACTOR REVERSIBLE SYSTEMS WITH APPLICATIONS TO PLANAR CENTERS

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Abstract. We present a generalization of the most usual symmetry in differential equations known as time-reversibility. We check that the typical properties are also valid for this new definition of reversibility. With it, we are able to present new families of planar polynomial vector fields having equilibrium points of center type. Moreover, we provide a new lower bound for the local cyclicity of polynomial vector fields of degree 6, $M(6) \geq 48$.

1. Introduction

One of the fundamental properties studied in natural science is the existence of symmetries. They appear usually in many physical models describing classical mechanics. The most important studied symmetry is known as the time-reversible one, being Birkhoff one of the first who used it. See, for example, his works on the restricted three-body problem studied in 1915 ([4]) or the billiard ball problem published in 1927 ([5]). There exists an extensive bibliography on symmetries and their properties in all areas of dynamical systems. See for example the nice survey of Lamb and Roberts published in 1998 ([18]). In particular, they describe how this time-symmetry is useful in mathematics and physics for understanding a big list of phenomena: symmetric periodic orbits, local bifurcations, homoclinic and heteroclinic orbits,... They appear also in other research branches as thermodynamics and quantum mechanics.

We recall the well-known definitions of two important symmetries for smooth vector fields: the reversible and the equivariant. Let $U \subset \mathbb{R}^n$ be an open set, $\varphi : U \to U$ be an involution of class $\mathcal{C}^1$, and $\mathcal{X} : U \to \mathbb{R}^n$ a vector field of class $\mathcal{C}^r$. We say that $\mathcal{X}$ is *φ-reversible* or *time-reversible with respect to φ* if

$$D\varphi \cdot \mathcal{X} = -\mathcal{X} \circ \varphi \tag{1}$$

and $\mathcal{X}$ is *φ-equivariant* if

$$D\varphi \cdot \mathcal{X} = \mathcal{X} \circ \varphi. \tag{2}$$

In both cases the phase portrait is symmetric with respect to the fixed points set

$$\operatorname{Fix} \varphi = \{x \in U \subset \mathbb{R}^n : \varphi(x) = x\}.$$

After Birkhoff we can quote the work of Devaney [10] where this definition is also used restricted to manifolds of even dimension $2n$ being n the dimension of the set $\operatorname{Fix} \varphi$. Some years later, Arnol'd and Sevryuk allow that the 'symmetry' φ is not to be necessarily an involution, see [1, 2].

The aim of this work is to extend the above definitions not only to treat them in a unified way but also to obtain new symmetric vector fields. Let $F : U \to \mathbb{R}$ a continuous function. Thus, we say that $\mathcal{X}$ is *F-factor φ-reversible* if

$$D\varphi \cdot \mathcal{X} = F\, \mathcal{X} \circ \varphi. \tag{3}$$

1991 *Mathematics Subject Classification.* Primary: 34C14, 34C25, 34C07.
Key words and phrases. Reversibility, involution, centers, local cyclicity, limit cycles.