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Generic Homeomorphisms with Shadowing of One-Dimensional Continua

Alfonso Artigue [1,*] and Gonzalo Cousillas [2,*]

1 Departamento de Matemática y Estadística del Litoral, Universidad de la República, Salto 50000, Uruguay
2 Instituto de Matemática y Estadística "Rafael Laguardia", Facultad de Ingeniería, Universidad de la República, Montevideo 11200, Uruguay
* Correspondence: artigue@unorte.edu.uy (A.A.); gcousillas@fing.edu.uy (G.C.)

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Abstract: In this article, we show that there are homeomorphisms of plane continua whose conjugacy class is residual and have the shadowing property.

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1. Introduction

Let (X, dist) be a compact metric space and denote by $\mathcal{H}(X)$ the space of homeomorphisms $f\colon X \to X$ with the C^0 distance

$$\text{dist}_{C^0}(f,g) = \sup\{\text{dist}(f(x), g(x)), \text{dist}(f^{-1}(x), g^{-1}(x)) : x \in X\}.$$

A property is said to be *generic* if it holds on a residual subset of $\mathcal{H}(X)$. Recall that a set is G_δ if it is a countable intersection of open sets and it is *residual* if it contains a dense G_δ subset. For instance, it is known that the shadowing property is generic for X a compact manifold ([1], Theorem 1) or a Cantor set ([2], Theorem 4.3). Recall that $f \in \mathcal{H}(X)$ has the *shadowing property* if for all $\varepsilon > 0$, there is $\delta > 0$ such that if $\{x_i\}_{i\in\mathbb{Z}}$ is a δ-pseudo orbit, then there is $y \in X$ such that $\text{dist}(f^i(y), x_i) < \varepsilon$ for all $i \in \mathbb{Z}$. We say that $\{x_i\}_{i\in\mathbb{Z}}$ is a *δ-pseudo orbit* if $\text{dist}(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$.

A remarkable result, proved in [3,4], states that if X is a Cantor set, then there is a homeomorphism of X whose conjugacy class is a dense G_δ subset of $\mathcal{H}(X)$. That is, a generic homeomorphism of a Cantor set is conjugate to this special homeomorphism. We say that $f, g \in \mathcal{H}(X)$ are *conjugate* if there is $h \in \mathcal{H}(X)$ such that $f \circ h = h \circ g$ and the *conjugacy class* of f is the set of all the homeomorphisms conjugate to f. This result gives rise to a natural question: besides the Cantor set,

which compact metric spaces have a G_δ dense conjugacy class?

On a space with a G_δ dense conjugacy class, the study of generic properties (invariant under conjugacy, as the shadowing property) is reduced to determine whether a representative of this class has the property or not.

In Theorem 2, we show that there are one-dimensional plane continua with a G_δ dense conjugacy class whose members have the shadowing property. The proof of this result is based on Theorem 1, where we show that for a compact interval I there is a G_δ conjugacy class in $\mathcal{H}(I)$ which is dense in the subset of orientation preserving homeomorphisms of I. In addition, the proof of Theorem 2 depends on Propositions 2 and 3, where we give sufficient conditions for the existence of a residual conjugacy class and for a homeomorphism to have the shadowing property, respectively. The following open question has an affirmative answer in the examples known by the authors:

if a homeomorphism has a G_δ dense conjugacy class, does it have the shadowing property?