ORBITALLY UNIVERSAL CENTERS

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Abstract. In this paper we define when a polynomial differential system is orbitally universal and we show the relevance of this notion in the classical center problem, i.e. in the problem of distinguishing between a focus and a center.

1. Introduction and statement of the main results

In this work we consider differential systems in $\mathbb{R}^2$ of the form

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

with P and Q polynomials having at the origin an isolated singular point. As usual the dot denotes derivative with respect to the time t. Along this paper we also consider the associated *vector field* $\mathcal{X} = P(x, y)\partial/\partial x + Q(x, y)\partial/\partial y$ to the differential system (1).

One of the main open problem in the qualitative theory of dynamical systems is to characterize when a singular point of system (1) has a center. This problem is known as the *center problem* and it consists in distinguishing between a center and a focus. A center is a singular point for which there exists a punctured neighborhood filled of periodic orbits, and a focus has a punctured neighborhood filled of spiraling orbits. We note that the center problem goes back to Poincaré [31] and Dulac [16].

There exist different algorithms to determine the necessary conditions to have a center when the linear part has purely imaginary eigenvalues, or it has zero eigenvalues but the linear part is not identically zero, see [4, 8, 18, 28, 31]. The characterization when the linear part is identically zero is a hard problem still open, see [20, 21, 22, 26, 29] for some partial results.

Another problem is to provide sufficient conditions in order that a singular point of the differential system (1) be center. Several mechanism are known, and some conjectures are established, see [27]. In this work we define one mechanism that provides the sufficiency of the center problem. This mechanism joint with the condition of the existence of a local analytic first integral contain all the known mechanisms for detecting centers up to now. Before define it we need to introduce some definitions and results.

For a monodromic singular point (i.e. for a focus or a center) the most important mechanism to have a center is to have a smooth first integral defined in a neighborhood of the singular point, see [30]. However to check the existence of this first

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