LIOUVILLIAN FIRST INTEGRALS FOR GENERALIZED RICCATI POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract. We characterize the generalized Riccati polynomial differential systems of the form $x' = y$, $y' = a(x)y^2+b(x)y+c(x)$, where $a(x)$, $b(x)$ and $c(x)$ are arbitrary polynomials that have a Liouvillian first integrals.

1. Introduction and statement of the main results

A classical problem in the qualitative theory of planar differential equations depending on parameters is to characterize the existence or non–existence of first integrals in function of these parameters.

Let x and y be complex variables. We consider the system

$$x' = y, \qquad y' = a(x)y^2 + b(x)y + c(x), \tag{1}$$

where $a(x)$, $b(x)$ and $c(x)$ are C^1 functions on x and the prime denotes derivative with respect to the time t that can be either real or complex. In fact, if $a(x)c(x) \not\equiv 0$ these systems are called *generalized Riccati differential systems*, if $a(x) \not\equiv 0$ and $c(x) \equiv 0$ they are *linear differential systems*, and if $a(x) \equiv 0$ they are *generalized Lienard differential systems*.

Our interest is on the *generalized Riccati polynomial differential systems* , i.e. when the functions $a(x)$, $b(x)$ and $c(x)$ are polynomials and we want to study its Liouvillian integrability.

The vector field associated to system (1) is

$$X = y\frac{\partial}{\partial x} + (a(x)y^2 + b(x)y + c(x))\frac{\partial}{\partial y}.$$

The main objectives of this paper is to characterize the Liouvillian first integrals of the generalized Ricatti polynomial differential systems.

Let $U \subset \mathbb{C}^2$ be an open set. We say that the non-constant function $H\colon U \to \mathbb{C}$ is a *first integral* of the polynomial vector field X on U if $H(x(t), y(t))$ is constant for all values of t for which the solution $(x(t), y(t))$ of X is defined on U. Clearly H is a first integral of X on U if and only if $XH = 0$ on U.

We recall that a *Liouvillian first integral* is a first integral H which is a Liouvillian function, that is, roughly speaking which can be obtained "by quadratures" of elementary functions. For a precise definition see [4]. The study of the Liouvillian first

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