

Applied Mathematical Sciences

Àlex Haro  
Marta Canadell  
Jordi-Lluís Figueras  
Alejandro Luque  
Josep-Maria Mondelo

# The Parameterization Method for Invariant Manifolds

From Rigorous Results to Effective  
Computations

 Springer

# Applied Mathematical Sciences

Volume 195

## Editors

S.S. Antman, Institute for Physical Science and Technology, University of Maryland, College Park, MD, USA

[ssa@math.umd.edu](mailto:ssa@math.umd.edu)

Leslie Greengard, Courant Institute of Mathematical Sciences, New York University, New York, NY, USA

[Greengard@cims.nyu.edu](mailto:Greengard@cims.nyu.edu)

P.J. Holmes, Department of Mechanical and Aerospace Engineering, Princeton University, Princeton, NJ, USA

[pholmes@math.princeton.edu](mailto:pholmes@math.princeton.edu)

## Advisors

J. Bell, Lawrence Berkeley National Lab, Center for Computational Sciences and Engineering, Berkeley, CA, USA

P. Constantin, Department of Mathematics, Princeton University, Princeton, NJ, USA

J. Keller, Department of Mathematics, Stanford University, Stanford, CA, USA

R. Kohn, Courant Institute of Mathematical Sciences, New York University, New York, USA

R. Pego, Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA, USA

L. Ryzhik, Department of Mathematics, Stanford University, Stanford, CA, USA

A. Singer, Department of Mathematics, Princeton University, Princeton, NJ, USA

A. Stevens, Department of Applied Mathematics, University of Münster, Münster, Germany

A. Stuart, Mathematics Institute, University of Warwick, Coventry, United Kingdom

S. Wright, Computer Sciences Department, University of Wisconsin, Madison, WI, USA

## Founding Editors

Fritz John, Joseph P. LaSalle and Lawrence Sirovich

More information about this series at <http://www.springer.com/series/34>



Àlex Haro • Marta Canadell • Jordi-Lluís Figueras  
Alejandro Luque • Josep-Maria Mondelo

# The Parameterization Method for Invariant Manifolds

From Rigorous Results to Effective  
Computations

 Springer

Àlex Haro  
Barcelona Graduate School of Mathematics  
Departament de Matemàtiques i Informàtica  
Universitat de Barcelona  
Barcelona, Spain

Marta Canadell  
Institute for Computational  
and Experimental Research  
in Mathematics  
Brown University  
Providence, USA

Jordi-Lluís Figueras  
Department of Mathematics  
Uppsala University  
Uppsala, Sweden

Alejandro Luque  
Instituto de Ciencias Matemáticas  
Consejo Superior de Investigaciones  
Científicas  
Madrid, Spain

Josep-Maria Mondelo  
Institut d'Estudis Espacials de Catalunya,  
and Barcelona Graduate School  
of Mathematics  
Departament de Matemàtiques  
Universitat Autònoma de Barcelona  
Bellaterra, Spain

ISSN 0066-5452

ISSN 2196-968X (electronic)

Applied Mathematical Sciences

ISBN 978-3-319-29660-9

ISBN 978-3-319-29662-3 (eBook)

DOI 10.1007/978-3-319-29662-3

Library of Congress Control Number: 2016933450

Mathematics Subject Classification (2010): 37-02, 37C55, 37J40, 34C45

© Springer International Publishing Switzerland 2016

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, express or implied, with respect to the material contained herein or for any errors or omissions that may have been made.

Printed on acid-free paper

This Springer imprint is published by Springer Nature  
The registered company is Springer International Publishing AG Switzerland

# Preface

Poincaré’s program for the global analysis of a dynamical system starts by considering simple solutions, such as equilibria and periodic orbits, together with their corresponding asymptotic solutions in forward and backward time. Geometrically speaking, these solutions correspond to invariant objects that form the skeleton of the dynamics in phase space. After the middle of the twentieth century, they were joined by a plethora of other invariant objects, such as hyperbolic sets, attractors, quasi-periodic orbits and normally hyperbolic invariant manifolds. All these landmarks were used to perform qualitative sketches to organize the long-term behavior of the system. With the advent of the age of computers, this qualitative approach started to be more quantitative, as researchers started to develop algorithms for effectively computing these invariant objects. Hence, it is not surprising that the last 30 years have witnessed a strong interest in the development of methods for their computation, spreading the range of applications and fostering the collaboration with other scientists and engineers. Meanwhile, the complexity of problems and applications has increased rapidly, thus motivating new research in the development of mathematical methods, computational algorithms and software implementations. Also, the interactions between these aspects have given rise to mutual refinements.

With the dawn of the twenty-first century, the parameterization method has emerged as a novel method that has promoted new developments in the theory and computation of invariant manifolds. It is a new point of view in which parameterizations of invariant manifolds are obtained through an analysis (which can be function-theoretical or numerical) of their invariance equations that takes advantage of the geometric structures of the problem under study. By its very nature, the parameterization method has led to a considerable synergy between rigorous mathematics and numerical computations. Of course, the methodology is not isolated and has received inspiration from many other approaches in each of the contexts it has been applied. Although traces of the method go back to Poincaré and particular formulations had been used in the literature, the systematic application of the method is relatively recent. The foundational papers of the parameterization method [CFdIL03a, CFdIL03b, CFdIL05] dealt with rigorous results on invariant manifolds

of fixed points of maps (some partial rigorous and numerical results had already appeared in [CF94, FR81, dIL97, Sim90]). The paper [dILGJV05] provided rigorous results on KAM theory without using classical angle-action coordinates (see some precedents in [dIL01]). The series of papers [HdIL06c, HdIL06b, HdIL07] considered invariant tori and whiskers of quasi-periodically forced systems, covering from rigorous results, numerical algorithms, and implementations in actual examples. Since then, the range of applications of the parameterization method has been continuously growing.

A remarkable property of the parameterization method is its applicability to different contexts in which other methodologies are fundamentally different. A first goal of this monograph is to provide a unified formulation of the parameterization method valid for different contexts. The specific contexts covered by this monograph are invariant manifolds associated with fixed points, invariant tori in quasi-periodically forced systems, invariant tori in Hamiltonian systems, and normally hyperbolic invariant manifolds. Although this plan may seem ambitious, our goal is not to provide a comprehensive treatment. Each of the contexts has a big amount of literature devoted to different theoretical and numerical techniques applicable. We will only cover the parameterization method, but not even in this case we will be comprehensive. For instance, the monograph is more focused in discrete than in continuous dynamical systems. Moreover, we do not cover the most recent results, because research on the parameterization method is still ongoing. This monograph complements the literature with new results, both rigorous and numerical, in contexts in which the parameterization method has already been applied. On the other hand, we also introduce normally hyperbolic invariant manifolds as a whole new context of application of the parameterization method.

The proofs done using the parameterization method involve proving convergence of iterative schemes that, by themselves, can be turned into numerical methods. This synergy between rigorous results and numerical methods is a signature of the parameterization method. A second goal of this monograph is to provide efficient and reliable algorithms for the numerical computation of invariant manifolds based on the parameterization method. Efficiency is attained through the use of the geometric structure of the problem, which leads to cancelations that simplify the structure of the functional equations to be solved at each iterative step. Reliability is a consequence of the proximity between algorithms and theory. For instance, error estimates for the approximate (numerically computed) parameterizations can be deduced easily from the invariance equations, and the non-degeneracy of the problem is usually a numerically evaluable hypothesis of the theorems that support the algorithms. In summary, we can obtain fast algorithms with low storage requirements and, more importantly, we have a notion of when they are reliable. Hence, it becomes possible to study an invariant object for parameter values very close to the one in which the object ceases to exist. These systematic studies lead to conjectures that enrich the theory.

A third objective of this monograph is to provide some methodology for computer-assisted proofs. The ability to produce theorems in a posteriori format is another

characteristic of the parameterization method. The rigorous numerical evaluation of the hypotheses of these theorems leads to a proof of the existence of a true invariant object near an approximate invariant one. A very convenient fact of this strategy is that the computer-assisted methodology is independent of the procedure (such as expansions, interpolation, or even hand calculations) used for the computation of the starting approximate invariant object.

A fundamental part of this monograph is a series of 12 fully detailed examples, some of which are computer-assisted proofs, that realize the three previous objectives. These examples are accompanied by some practical details of their implementation, so that the reader can either reproduce them or adapt the methodology to other problems. A public version of the software used for some of these examples is available at <http://www.maia.ub.es/dsg/param/>.

The parameterization method is unique in its ability to be applied to a problem in several stages, all mentioned in the previous paragraphs, that go from rigorous results to validated numerical results. These stages give rise to the following program: write the functional equations for the parameterization of an invariant object (the invariance equations), provide adequate functional frameworks to ensure the convergence of iterative methods for the solution of these equations, to develop numerical algorithms based on these iterative schemes, implement them in actual problems using appropriate discretizations, and rigorously validate (invoking an a posteriori theorem) the numerical results. This “from theory-to algorithms-to computations-to validations” philosophy is a driving force in this monograph.

We believe that several types of readers can benefit from this monograph. It is aimed to either applied scientists and engineers with an interest in rigorous developments or more theoretically oriented mathematicians with an interest in applications. For instance, a reader interested in the implementation of the parameterization method in applications can benefit from the detailed algorithmic descriptions of this monograph. A more mathematically oriented reader interested in KAM theory can find a complete proof of a KAM theorem in a posteriori format. The theoretical and algorithmic parts are self-contained and can be read independently.

The reader is assumed to have some familiarity with dynamical systems, more particularly with invariant manifolds and normal forms. A reader novel to dynamical system can consult introductory books such as [Arn88, BS02, Chi06, Irw01, KH95, HK03, GH90, PdM82, Rob95, Rob04]. Except for this fact, this monograph is essentially self-contained. It is divided in 5 chapters, of which the first one is an introduction and the remaining ones correspond to different contexts of application of the parameterization method. Except for notation drawn from the first chapter, Chapters 2 to 5 are independent of each other.

Chapter 1 starts by providing an overview of the literature. After that, it introduces unified formulations of the parameterization method for invariant manifolds of fixed points and for invariant tori in different contexts. These formulations are the basis of the subsequent chapters. This chapter can be considered a reading guide of the rest of the book.



Chapter 2 discusses computational aspects of invariant manifolds of vector fields at fixed points. It is focused on algorithms and implementations, since the theory of invariant manifolds of fixed points is well established. There are many classical textbooks including the main results of the theory, to which the trilogy [CFdIL03a, CFdIL03b, CFdIL05] adds the rigorous results of the parameterization method. The goal is to provide algorithms for the computation of semi-local expansions, based on the algebraic manipulation of power series and novel automatic differentiation techniques. The detailed examples of this chapter are the 2D stable manifold of the origin of the Lorenz system, the 4D center manifold of a collinear point of the Restricted Three-Body Problem, and a 6D partial normal form in the same problem that allows the generation of Conley’s transit and non-transit trajectories associated with any object of the center manifold.

Chapter 3 revisits the papers [HdIL06c, HdIL06b, HdIL07, FH12]. First, it provides a full proof of a Kantorovich-like theorem for invariant tori in discrete quasi-periodic systems. The proof of this theorem leads to several algorithms for the computation of invariant tori in this context that are also detailed. Next, we explain a computer-assisted methodology for the validation of numerical results based on the previous a posteriori theorem. The chapter ends with three examples: validation of saddle invariant tori on the verge of breakdown, computation of a rigorous upper bound of the measure of Cantor-like spectra of a discrete Schrödinger operator, and validation of an attracting torus that by direct double precision seems to be a strange nonchaotic attractor.

Chapter 4 is devoted to the parameterization method in KAM theory, also referred to as KAM theory without action-angle coordinates. It adds a more geometrical perspective to the original paper [dILGJV05] in the spirit of [GHdIL14]. More broad views on KAM theory can be found in [BHS96, dIL01], which include many references to the extensive literature. The chapter states and proves a KAM theorem in a posteriori format, with explicit bounds suitable to be applied in an effective and quantitative way. The proof is quite technical, but the reader can skip it without losing the flavor of the application of the method. We have included full descriptions of the derived algorithms and applications to the examples that follow, which are application of the theorem (by hand calculations) to obtain persistence of the golden invariant curve for tiny values of the parameter of the standard map, numerical continuation of this same curve up to values close to breakdown, and computation of 2D tori in the Froeschlé map.

Chapter 5 presents some ideas of normally hyperbolic manifold theory, focusing on the algorithmic application of the parameterization method in such context (the classical theory can be found in [HPS77, Fen72], and a more recent account in [Wig94]). This new method is applied to the following examples: computation of an attracting invariant curve in a 2D Fattened Arnold family, computation of a saddle invariant curve in a 3D Fattened Arnold family, and the computation of a 2D normally hyperbolic invariant cylinder in the Froeschlé map.

Along the monograph, we cover all the aspects of the “from theory-to algorithms-to computations-to validations” program, although not all the aspects are covered in

each chapter. Chapter 2 focuses on algorithmic and practical issues on the computation of invariant manifolds of fixed points. Chapter 3 covers the full program for a particular case (invariant tori in quasi-periodic systems). Chapter 4 is close to that, since it covers the first three aspects, and the KAM theorem stated there is ready to be used in computer-assisted proofs. Chapter 5 covers new research on the parameterization method for normally hyperbolic invariant manifolds, in particular on development of numerical algorithms. We emphasize these and other novelties in Chapter 1.

We finish this preface paraphrasing the following inspiring words in the review [CDD<sup>+</sup>91], written by S. Coffey, A. Deprit, E. Deprit, L. Healy, and B. R. Miller more than 20 years ago: *“The discipline (of nonlinear dynamics) instead must try with tenacity to keep pace with computational technology and make room for its innovations the same way. The challenge thus is endless, for each generation of mathematical physicist needs to keep abreast of techniques relentlessly emerging from the engineering shops.”* And techniques emerge not only from the engineering shops but also from the rigorous results in mathematical papers. Hence, researchers benefit from the combination and feedback between theorems, algorithms, and numerical experiments that often spur conjectures that motivate further research. The parameterization method is one of the emerging techniques in the area of dynamical systems. The research is on the way, and there is still much to come.

## Acknowledgments

We would like to acknowledge the financial support we have received during these years from different sources: M.C. from the Spanish grants MTM2009-09723, MTM2012-32541 and MTM2015-67724-P, the FPI grant BES-2010-039663, the Catalan grant 2014-SGR-1145, and the NSF grant DMS-1500943; J.-Ll.F. from the Spanish grants MTM2009-09723, MTM2012-32541, and the Catalan grant 2009-SGR-67; A.H. from the Spanish grants MTM2009-09723, MTM2012-32541 and MTM2015-67724-P, and the Catalan grants 2009-SGR-67 and 2014-SGR-1145; A.L. acknowledges support from postdoctoral positions in the Juan de la Cierva Fellowship JCI-2010-06517 (years from 2012 to 2014) and in the ERC Starting grant 335079 (from 2015), the Spanish grant MTM2012-32541, the ICMAT-Severo Ochoa grant SEV-2015-0554 (MINECO), and the Catalan grants 2009-SGR-859 and 2014-SGR-1145; and J.-M.M. from the Spanish grants MTM2011-26995-C02-01, MTM2010-16425 and MTM2014-52209-C2-1-P, and the Catalan grant 2009-SGR-410.

Our exposition includes contributions of many researchers, to whom we would like to express our appreciation. We owe Xavier Cabré, Ernest Fontich, Alejandra González, Àngel Jorba, Jordi Villanueva, and, very specially, Rafael de la Llave, the major contributions that laid the foundations for the parameterization method in several contexts of dynamical systems.

Our gratitude is extended to the members of the Dynamical Systems group in Barcelona and especially to its founder, Carles Simó. Carles has always been a source of inspiration and encouragement for us. It is thanks to him that the Barcelona group has become what it is today, with a large number of researchers working in many areas of dynamical systems, and with emphasis on applications and numerical calculations. This environment has been instrumental in our training as mathematicians and in our research. We would like also to thank the research and training activities promoted by the Spanish DANCE network.

It is our pleasure to acknowledge all those who helped us in different drafts of this book. Thanks go to Renato Calleja, Ernest Fontich, Alejandra González, Zubin Olikara, Carles Simó, Arturo Vieiro, and Chongchun Zeng, for careful reading of parts of the manuscript and for giving us many valuable comments and suggestions. The comments and suggestions made by anonymous referees have been also relevant for improving the material. Aleix Boquet, Yu Chen, Carlos Domingo and Albert Granados have also read parts of the manuscript and found several mistakes. Of course, the errors that remain are our responsibility.

We also wish to thank fruitful discussions with many other researchers and colleagues during the last years, which have enriched our knowledge on the topic of dynamical systems. Among them, we mention Luís Benet, Henk Broer, Maciej Capiński, Luca Dieci, George Haller, Gemma Huguet, Bernd Krauskopf, Martin Lo, Jay Mireles-James, Hinke Osinga, Daniel Peralta-Salas, Joaquim Puig, Frank Schilder, Mikhail Sevryuk, and Piotr Zgliczyński.

We would like to thank our editor, Achi Dosanjh, for her kindness, enthusiasm, dedication, and patience during the whole process. Since Rafael de la Llave introduced us in the AIMS conference in Orlando (2012), the project evolved from being a relatively long review to a thick volume thanks to Achi's support. The acknowledgment is extended to Springer-Verlag for supporting this project and to SPi Technologies for handling the production of the book.

We are grateful to all the people and institutions that have organized scientific events (conferences and courses) in which we have presented part of the preliminary material of this book: DANCE coordinators Lluís Alsedà, Amadeu Delshams, Àngel Jorba, Carmen Núñez, and Enrique Ponce, DANCE network (RTNS 2004, DDays 2014); Àngel Jorba and Carles Simó, IMUB, DANCE network, and i-Math consolider project (advanced course on specific algebraic manipulators, 2007; i-Math doc-course on computational methods in dynamical systems and applications, 2010); and Peter Bates and Rafael de la Llave, the IMA (IMA New Directions Short Course: Invariant Objects in Dynamical Systems and their Applications, 2012). We also thank the many institutions that have hosted us in several research visits, especially University of Texas at Austin, Uppsala University, and Georgia Institute of Technology.

We would like to express again our gratitude to Rafael de la Llave, from whom we continuously learn about the mysteries of the parameterization method, and many other aspects of dynamical systems. Rafa has continuously encouraged us to accomplish the task of writing this book. Our gratitude goes well beyond science: we thank him for his friendship, encouragement, and support along our lives.

And finally, but most especially, our sincere gratitude goes to our families and friends, for their encouragement, support, patience, and the many hours that we could not spend together throughout this long, but very exciting, project.

Barcelona, Spain  
Providence, RI, USA  
Uppsala, Sweden  
Madrid, Spain  
Bellaterra, Spain  
March 2016

Àlex Haro  
Marta Canadell  
Jordi-Lluís Figueras  
Alejandro Luque  
Josep-Maria Mondelo



# Contents

<b>1</b>	<b>An Overview of the Parameterization Method for Invariant Manifolds</b>	<b>1</b>
1.1	Historical Overview	1
1.2	Parameterizations of Invariant Manifolds	8
1.2.1	Invariance Equations for Invariant Manifolds	8
1.2.2	Styles of Parameterizations	9
1.3	Invariant Manifolds of Fixed Points for Diffeomorphisms	11
1.3.1	The Invariance Equation	11
1.3.2	The Cohomological Equations	13
1.3.3	Styles of Parameterizations	15
1.3.4	A Few Words on Implementations	18
1.4	A Meta-Algorithm for Computing Invariant Tori	18
1.4.1	Geometric Setting	19
1.4.2	Newton's Method and Approximate Reducibility	20
1.4.3	Some Specialized Algorithms	22
1.4.4	A few more Words on Implementations	27
<b>2</b>	<b>Seminumerical Algorithms for Computing Invariant Manifolds of Vector Fields at Fixed Points</b>	<b>29</b>
2.1	From Power Series to Automatic Differentiation	30
2.2	Computation of Invariant Manifolds and Normal Forms	32
2.2.1	The Invariance Equation	33
2.2.2	The Cohomological Equations	34
2.2.3	Styles of Parameterizations	36
2.2.4	Complexification and Realification	37
2.2.5	Error Estimates	38
2.3	An Algebraic Manipulator of Multivariate Power Series	39
2.3.1	The Algebra of Power Series	40
2.3.2	Elementary Functions of Power Series	40
2.3.3	A Working Definition of Algorithmic Complexity	42
2.3.4	Some Implementation Details	45

2.4	Example 1: The Lorenz Manifold	46
2.4.1	Expansions and Fundamental Domain	46
2.4.2	Globalization and Error Estimation	48
2.5	Example 2: The Center Manifold of the $L_1$ Point in the Earth-Moon Circular, Spatial Restricted Three-Body Problem	53
2.5.1	A Brief Description of the RTBP	53
2.5.2	Computation of the Center Manifold as a Graph	55
2.5.3	Growth of the Coefficients of the Center Manifold	56
2.5.4	Dynamics on an Energy Level in the Center Manifold	56
2.5.5	Computation of the Center Manifold Using a Mixed Style	61
2.6	Example 3: Partial Normal Forms and Transit Trajectories	65
2.6.1	Transit Trajectories	67
2.6.2	Expansions and Error Estimation	68
2.6.3	Some Sample Trajectories	71
<b>3</b>	<b>The Parameterization Method for Quasi-Periodic Systems: From Rigorous Results to Validated Numerics</b>	<b>75</b>
3.1	Robustness, Hyperbolicity, Computability, and Validity	76
3.2	Skew-product Systems and Fiberwise Hyperbolic Invariant Graphs	79
3.3	Two Validation Theorems	82
3.3.1	A Kantorovitch-type Theorem	82
3.3.2	Approximate Reducibility, or How to Verify the Hyperbolicity Condition	85
3.4	Computation of Invariant Tori	88
3.4.1	Large Matrix Methods	88
3.4.2	Projection Method	90
3.4.3	Reducibility Method	93
3.5	Implementation of the Validation Algorithm	97
3.5.1	Validated Numerics and Interval Arithmetic	97
3.5.2	Fourier Models	99
3.5.3	A Validation Algorithm	102
3.5.4	Some Interval Arithmetic Software Packages	104
3.6	Example 4: Saddle Tori on the Verge of the Hyperbolicity Breakdown	104
3.6.1	Numerical Computation of Invariant Tori	104
3.6.2	Computer Validations	106
3.7	Example 5: Rigorous Upper Bounds of Spectra of Discrete Quasi-Periodic Schrödinger Operators	109
3.7.1	The Harper Map	109
3.7.2	Computer Validations	110
3.8	Example 6: Rediscovering a Fake Strange Nonchaotic Attractor	113
3.8.1	A non-Strange Nonchaotic Attractor	114
3.8.2	Computer Validations	115

- 4 The Parameterization Method in KAM Theory** ..... 119
  - 4.1 Existence and Persistence of Quasi-Periodic Motions ..... 120
  - 4.2 Geometric Properties of Invariant Tori ..... 124
    - 4.2.1 Symplectic Structures on the Annulus and Lagrangian Tori . 124
    - 4.2.2 Construction of Symplectic Adapted Frames ..... 126
  - 4.3 A KAM Theorem for Exact Symplectic Maps ..... 130
    - 4.3.1 Cohomological Equations ..... 131
    - 4.3.2 Approximately Invariant Tori and Approximate Reducibility ..... 131
    - 4.3.3 Analytic Preliminaries, Norms, and Small Divisors ..... 137
    - 4.3.4 The KAM Theorem ..... 140
    - 4.3.5 One Step of the Newton-like Method ..... 142
    - 4.3.6 Convergence of the KAM Process ..... 148
  - 4.4 Example 7: Application of the KAM Theorem to the Standard Map ..... 151
    - 4.4.1 Using the Approximation Given by the Planar Torus ..... 152
    - 4.4.2 Using the Approximation Given by Higher Order Lindstedt Series ..... 157
  - 4.5 An Algorithm to Compute Invariant Tori ..... 162
  - 4.6 Example 8: Continuation of the Golden Curve in the Standard Map . 168
  - 4.7 Example 9: Continuation of Invariant Tori in the Froeschlé Map ... 172
  - 4.8 Other Remarks and Generalizations ..... 175
    - 4.8.1 Translated Tori Theorems and Non-Twist Tori ..... 177
    - 4.8.2 Hamiltonian Systems ..... 178
    - 4.8.3 Lower Dimensional (Isotropic) Tori ..... 181
    - 4.8.4 Invariant Tori in Dissipative Systems ..... 184
  
- 5 A Newton-like Method for Computing Normally Hyperbolic Invariant Tori** ..... 187
  - 5.1 On the Numerical Computation of Normally Hyperbolic Invariant Manifolds ..... 188
  - 5.2 Normally Hyperbolic Invariant Tori ..... 191
    - 5.2.1 Normal Hyperbolicity ..... 192
    - 5.2.2 Invariant Tori and Adapted Frames ..... 193
  - 5.3 Specification of one Step of a Newton-like Method ..... 196
    - 5.3.1 Substep 1: Correction of the Approximate Invariant Torus .. 196
    - 5.3.2 Substep 2: Correction of the Stable and Unstable Bundles ..... 199
    - 5.3.3 Substep 3: Computation of Approximate Inverses ..... 201
    - 5.3.4 A Continuation Method ..... 202
  - 5.4 Some Guidelines for the Implementations ..... 203
    - 5.4.1 Modeling Tori ..... 203
    - 5.4.2 Manipulation of Functions ..... 205
    - 5.4.3 Grid Point Methods Versus Spectral Methods: a Digression . 207



- 5.5 Example 10: Continuation of Attracting Tori in a 2D-Fattened Arnold Family ..... 208
  - 5.5.1 The Unperturbed Case ..... 209
  - 5.5.2 Computations far from the Perturbative Regime ..... 210
- 5.6 Example 11: Continuation of Saddle Tori in a 3D-Fattened Arnold Family ..... 217
  - 5.6.1 The Unperturbed Case ..... 219
  - 5.6.2 Continuation far from the Perturbative Regime ..... 220
  - 5.6.3 Continuation Starting Close to the Main Resonance ..... 225
- 5.7 Example 12: Computation of a Normally Hyperbolic Invariant Cylinder ..... 230
  
- References** ..... 239
  
- Index** ..... 259